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AN ALGEBRAIC PROOF THAT $[\Omega^U]_2 = \mathfrak{N}^2$

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ABSTRACT. It is often desirable to find the image of one cobordism theory in another. Milnor settled the first problem of this sort, the image of complex cobordism in unoriented cobordism, by construction of concrete generating manifolds. When such constructions are too difficult, it still may be possible to solve the problem using more algebraic methods. This note offers a proof of Milnor's result which depends on the Adams spectral sequence and requires no ad hoc construction of manifolds.

1. **Introduction.** Milnor [7] proved the following

THEOREM. *The image of the complex cobordism ring Ω^U in the unoriented cobordism ring $\mathfrak{N}$ is the subring $\mathfrak{N}^2$ of all squares of elements of $\mathfrak{N}$.*

Moreover Stong [9, p. 138] showed that generators b_i for the polynomial ring Ω^U and x_i for the polynomial ring $\mathfrak{N}$ can be chosen such that

$$\begin{aligned} f_\#(b_i) &= (x_i)^2, \quad i \neq 2^a - 1, \\ &= 0, \qquad\quad i = 2^a - 1, \end{aligned}$$

where $f_\# : \Omega^U \to \mathfrak{N}$ is the obvious map. Both these proofs rely on constructing manifolds which generate the cobordism rings. This note presents an algebraic proof of the refined theorem, based on the naturality of the Adams spectral sequence.

In general, for closed subgroups G, H of O, $G \subset H$, there is the map $\Omega^G \to \Omega^H$, which one would like to describe. Constructions like those of Milnor and Stong seem very difficult to apply to these problems. However, an algebraic method should apply to such situations even in the absence of a plethora of examples.

2. **Notation and preliminaries.** Cohomology groups will be mod 2 unless otherwise indicated. Recall the standard description of $H^*(BO)$ as the Z_2 vector space with basis $\{w_\omega\}$ where ω runs through all partitions

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$(i_1, \cdots, i_n)$, and if $|\omega| = i_1 + \cdots + i_n$ then $w_\omega \in H^{|\omega|}(BO)$. Of course $w_\omega = s_\omega(w_1, w_2, \cdots, w_N)$, where w_i is the ith universal Stiefel-Whitney class, and s_ω is the polynomial expressing the symmetric function $\sum t_1^{i_1} \cdots t_n^{i_n}$ in terms of the elementary symmetric functions $\sigma_j = \sum t_1 \cdots t_j$,

$$s_\omega(\sigma_1, \cdots, \sigma_N) = \sum t_1^{i_1} \cdots t_n^{i_n}.$$

Similarly $H^*(BU)$ has a Z_2-basis $\{c_\omega\}$, where $c_\omega \in H^{2|\omega|}(BU)$, and $c_\omega = s_\omega(c_1, \cdots, c_N)$, the c_i being the mod 2 reduction of the ith Chern class.

As usual we have the Thom spectra $\boldsymbol{MO}$ and $\boldsymbol{MU}$, and the Thom isomorphisms $\Phi: H^*(BO) \to H^*(\boldsymbol{MO})$, $\Phi: H^*(BU) \to H^*(\boldsymbol{MU})$ given by $\Phi(x) = x \cdot \mathrm{U}$, where U is the stable Thom class in $H^0(\boldsymbol{MO})$ or $H^0(\boldsymbol{MU})$, respectively. Now we recall Thom's Theorem (cf. [**4**]) that $H^*(\boldsymbol{MO})$ is a free module over $\mathscr{A}$, the Steenrod algebra, on generators

$$\{w_\omega \cdot \mathrm{U} : \omega = (i_1, \cdots, i_n), i_j \neq 2^a - 1\}.$$

Similarly Milnor proved [**6**] that as an $\mathscr{A}$-module, $H^*(\boldsymbol{MU})$ is a free $\mathscr{A}/(Q_0)$-module on generators $\{c_\omega \cdot \mathrm{U} : \omega = (i_1, \cdots, i_n), i_j \neq 2^a - 1\}$.

Next recall that the Whitney sum defines a map $BG \times BG \to BG$, and thus a map $\boldsymbol{MG} \wedge \boldsymbol{MG} \to \boldsymbol{MG}$, making $\boldsymbol{MG}$ a ring spectrum. Thus we have a diagonal map $\Psi: H^*(\boldsymbol{MG}) \to H^*(\boldsymbol{MG}) \otimes H^*(\boldsymbol{MG})$. From the definition of these diagonals it is clear that the Whitney sum formula gives

$$\Psi(w_\omega \cdot \mathrm{U}) = \sum_{\omega = (\omega_1, \omega_2)} (w_{\omega_1} \cdot \mathrm{U}) \otimes (w_{\omega_2} \cdot \mathrm{U})$$

and similarly for $\boldsymbol{MU}$. At this point we have described $H^*(\boldsymbol{MO})$ as an $\mathscr{A}$-coalgebra, and $H^*(\boldsymbol{MU})$ as both an $\mathscr{A}$-module and a Z_2-coalgebra (we will not need the $\mathscr{A}$-coalgebra structure of $H^*(\boldsymbol{MU})$).

Regarding the universal $\boldsymbol{C}^n$-bundle over $BU(n)$ as an $\boldsymbol{R}^{2n}$-bundle defines a map $BU(n) \to BO(2n)$. In the limit, we have $f: BU \to BO$ which induces $f: \boldsymbol{MU} \to \boldsymbol{MO}$ and thus $f_\#: \Omega^U \to \mathfrak{N}$.

PROPOSITION 1. *The map $f^*: H^*(BO) \to H^*(BU)$ is given by*

$$f^*(w_{(\omega,\omega)}) = c_\omega, \qquad f^*(w_\alpha) = 0, \quad \alpha \neq (\omega, \omega).$$

PROOF. Let ξ be a complex bundle, and let $r(\xi)$ be the underlying real bundle. If $c_i(\xi)$ is the mod 2 reduction of the ith Chern class of ξ, then $w_{2i}(r(\xi)) = c_i(\xi)$, $w_{2i+1}(r(\xi)) = 0$ [**9**, p. 74]. Let $g: BO \to BU$ classify the complexification of the universal bundle γ. Then $f \circ g$ classifies $\gamma \oplus \gamma$, and the Whitney sum formula shows that $(f \circ g)^* w_{2i} = w_i^2$, $(f \circ g)^* w_{2i+1} = 0$. Thus g^* is a monomorphism, and $g^*(c_i) = w_i^2$. Now the Thom product formula shows [**9**, pp. 71, 256] that $(f \circ g)^* w_{\omega,\omega} = (w_\omega)^2$ and $(f \circ g)^* w_\alpha = 0$, $\alpha \neq (\omega, \omega)$. Finally, $g^*(c_\omega) = (w_\omega)^2$, since squaring is a homomorphism in a polynomial ring over Z_2, so the proposition follows.

3. **Proof of the theorem.** The theorem will be proved by tracing through the maps induced by f in the Adams spectral sequence. Since MO and MU are ring spectra, the diagonals in $H^*(MO)$ and $H^*(MU)$ induce products in the E_2 terms which project to the products in E_∞ induced by the products in $\mathfrak{N}$ and Ω^U. We begin by constructing minimal $\mathscr{A}$-resolutions $H^*(MO) \leftarrow CO$ and $H^*(MU) \leftarrow CU$. Let CO_0 be the free $\mathscr{A}$-module on $\{\tilde{w}_\omega : \omega = (i_1, \cdots, i_n), i_j \neq 2^a - 1\}$, and define $\varepsilon : CO_0 \to H^*(MO)$ by $\varepsilon(\tilde{w}_\omega) = w_\omega \cdot \mathrm{U}$. Let CU_0 be the free $\mathscr{A}$-module on $\{\tilde{c}_\omega : \omega = i_1, \cdots, i_n), i_j \neq 2^a - 1\}$, and define $\varepsilon : CU_0 \to H^*(MU)$ by $\varepsilon(\tilde{c}_\omega) = c_\omega \cdot \mathrm{U}$. Note that the kernel of ε is contained in the augmentation submodule, so ε may be extended to a minimal resolution $H^*(MU) \leftarrow CU$. Then we have $\mathscr{A}$-resolutions

$$\begin{array}{ccccccccc} 0 & \longleftarrow & H^*(MO) & \overset{\varepsilon}{\longleftarrow} & CO_0 & \longleftarrow & 0 & & \\ & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f_0} & & & & \\ 0 & \longleftarrow & H^*(MU) & \overset{\varepsilon}{\longleftarrow} & CU_0 & \longleftarrow & CU_1 & \longleftarrow & \cdots . \end{array}$$

Here $\{f_i\} : CO \to CU$ is the chain map, unique up to homotopy, lifting f^*.

Let $C^*G_i = \mathrm{Hom}_{\mathscr{A}}(CG_i, Z_2)$, $G = O, U$. Then $\{C^*G_i\}$, with the obvious maps, forms a chain complex whose homology is by definition $\mathrm{Ext}_{\mathscr{A}}(H^*(MG); Z_2)$. By the minimality of the resolutions, these maps are zero, and we have $\mathrm{Ext}_{\mathscr{A}}^{s,t}(H^*(MG); Z_2) = \mathrm{Hom}_{\mathscr{A}}^t(CG_s; Z_2) = (C^*G_s)_t$. The diagonal maps in $H^*(MG)$ lift to diagonal maps in CG_0 and thus give products in C^*G_0. Now C^*O_0 is the Z_2-vector space with basis $\{w_\omega^* : \omega = (i_1, \cdots, i_n), i_j \neq 2^a - 1\}$, where w_ω^* is dual to $\tilde{w}_\omega$. Similarly C^*U_0 is the Z_2-vector space on the duals c_ω^* of $\tilde{c}_\omega$. Also $\{f_i\} : CO \to CU$ defines $\{f_i^*\} : C^*U \to C^*O$.

PROPOSITION 2. *The map $f_0^* : C^*U_0 \to C^*O_0$ is given by $f_0^*(c_\omega^*) = (w_\omega^*)^2$.*

PROOF. By Proposition 1, we have $f_0^*(c_\omega^*) = w_{\omega,\omega}^*$. The formula for Ψ shows that $(w_\omega \cdot \mathrm{U}) \otimes (w_\omega \cdot \mathrm{U})$ occurs in $\Psi(w_\alpha \cdot \mathrm{U})$ only for $\alpha = (\omega, \omega)$, so $w_{\omega,\omega}^* = (w_\omega^*)^2$.

Thus we have a formula for $\{f_i^*\} : C^*U \to C^*O$, since C^*O_i and f_i^* are zero for $i > 0$. This is the map on the E_2 level of the Adams spectral sequences for Ω^U and $\mathfrak{N}$, since the product in C^*G_0 coincides with the product in E_2. Clearly the differentials vanish in the MO spectral sequence, and Milnor shows [6] they also vanish for MU, so the f_0^* determines the map at the E_∞ level. Now the filtration of $\mathfrak{N} = \pi_*(MO)$ guaranteed by the spectral sequence is certainly trivial. Thus the generator $x_i \in \mathfrak{N}_i$ must be represented by some $v_i \in (C^*O_0)_i$, and Liulevicius shows [4] that $v_i = w_{(i)}^* +$ decomposables. In the MU case we have a filtration $\pi_j(MU) = F_j^0 \supset F_j^1 \supset \cdots$, with $(C^*U_0)_j = F_j^0 / F_j^1$. Recall [6], [9] that Ω^U is a polynomial ring on generators $b_i \in \Omega_{2i}^U$. Then the calculation [5, p. 13] of

$\mathrm{Ext}_{\mathscr{A}}(H^*(MU), Z_2)$ shows that b_i, $i=2^a-1$, is represented by an element of filtration one, so $f_\#(b_{2^a-1})=0$. Let e_i be the projection of $b_i \in \Omega_{2i}^U = F_{2i}^0$ in $(C^*U_0)_{2i}$. Then again $e_i = c_{(i)}^* +$decomposables for $i \neq 2^a-1$, since the diagonal in CU_0 is just like that in CO_0. The refined theorem now follows from Proposition 2.

4. **Remarks.** Of course this procedure works equally well with the Adams spectral sequence for general cohomology theories. In particular Novikov [**8**] has shown that the computation of Ω^{SU} [**1**], [**2**] can be considered as an application of the spectral sequence in U^*-theory. In the same sense, it seems likely that the determination of $[\Omega^{SU}]_2$ by Conner and Landweber [**3**] can be considered an application of the above process for the U^*-theory spectral sequence.

There is a cobordism theory of manifolds with stable tangent bundle equal to a complexification of a complex bundle. Such manifolds have an obvious symplectic structure. The situation is summarized by the inclusions $U \otimes C \subset \mathrm{Sp} \subset 0$. The only Sp-manifolds constructed thus far in the literature are $U \otimes C$-manifolds (cf. [**10**]). Applying the mod 2 Adams spectral sequence we see there must be an element in Ω_5^{Sp} which admits no $U \otimes C$ structure. Nigel Ray reports that he has constructed this manifold, as well as other such examples.

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