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CYCLIC VECTORS OF INDUCED REPRESENTATIONS

A. HULANICKI AND T. PYTLIK

In this note we prove that for a first countable locally compact group every unitary representation induced by a cyclic representation is cyclic.

This result has been recently obtained also by F. Greenleaf and M. Moskowitz [2] in a more complicated way.

Let G be a first countable locally compact group. Let $\mathscr{K}$ be the space of continuous functions with compact support equipped with the Schwartz topology on G. Let D be the cone in $\mathscr{K}'$ of positive-definite measures on G, i.e., $\mu \in D$ if $\langle x^* * x, \mu \rangle \geqq 0$ for all x in $\mathscr{K}$. For each μ in D we define $I = \{x \in \mathscr{K} : \langle x^* * x, \mu \rangle = 0\}$. Then $\mathscr{H}^0_\mu = \mathscr{K}/I$ is a pre-Hilbert space with a strictly positive-definite inner product $(\bar{x}, \bar{y})_\mu = \langle y^* * x, \mu \rangle$, where $x \to \bar{x}$ is the natural mapping of $\mathscr{K}$ onto $\mathscr{H}^0_\mu$. Moreover, if $L_g x(h) = x(g^{-1}h)$, $g, h \in G$, then I is stable under L_g, $g \in G$, and so L_g acts on $\mathscr{H}^0_\mu$ and is unitary with respect to $(\cdot, \cdot)_\mu$. As such it extends to the completion $\mathscr{H}_\mu$ of $\mathscr{H}^0_\mu$. Let $g \to L^\mu_g$ be the representation thus obtained. If P is a projection in $\mathscr{H}_\mu$ which commutes with the L^μ_g, $g \in G$, then there exists a unique measure ν in D such that

$$(*) \qquad (P\bar{x}, \bar{y})_\mu = \langle y^* * x, \nu \rangle \quad \text{for all } x, y \text{ in } \mathscr{K}.$$

For the details and proofs cf. [1, pp. 40–48].

Let $\{e_n\}$, $n = 1, 2, 3, \cdots$, be an approximate unit in $\mathscr{K}$ and let

$$\xi = \sum_1^\infty \lambda_n e_n^* * e_n,$$

where the λ_n are positive and such that the series is uniformly convergent.

THEOREM. *For every measure μ in D, the vector ξ is cyclic with respect to L^μ_g in $\mathscr{H}_\mu$.*

PROOF. Suppose that ξ belongs to a subspace $\mathscr{N}$ of $\mathscr{H}_\mu$ which is L^μ_g invariant, $g \in G$. Let P be a projection onto $\mathscr{N}^\perp$. Clearly, P commutes with L^μ_g and so it defines a measure ν in D such that $(*)$ holds. Since

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$P\bar{\xi}=0$, for every e_n, $n=1, 2, 3, \cdots$, we have $0=(P\bar{\xi}, \bar{e}_n)=\langle e_n^* * \xi, \nu\rangle$, whence

$$\langle \xi, \nu\rangle = 0 \quad \text{and} \quad \sum_1^\infty \lambda_n(e_n^* * e_n, \nu) = 0.$$

Thus, since $\lambda_n>0$ and $\langle e_n^* * e_n, \nu\rangle \geqq 0$, we have $\langle e_n^* * e_n, \nu\rangle=0$ for all $n = 1, 2, 3, \cdots$, which, by Schwartz inequality, means that $\langle e_n * x, \nu\rangle=0$ for all x in $\mathscr{K}$ and $n=1, 2, 3, \cdots$, so $\nu=0$, which implies that $P=0$ and $\mathscr{N}=\mathscr{H}_\mu$.

Since every unitary representation induced from a cyclic representation is of the form $g\rightarrow L_g^\mu$ for a positive-definite measure μ (cf. [**1**]) we have

COROLLARY. *For a first countable locally compact group every unitary representation induced by a cyclic representation is cyclic.*

REFERENCES

1. E. Effros and F. Hahn, *Locally compact transformation groups and C*-algebras*, Mem. Amer. Math. Soc. No. 75 (1967). MR **37** #2895.

2. F. Greenleaf and M. Moskowitz, *Cyclic vectors for representations of locally compact groups* (to appear).

INSTITUTE OF MATHEMATICS, THE POLISH ACADEMY OF SCIENCES, WARSAW, POLAND

INSTITUTE OF MATHEMATICS, UNIVERSITY OF WROCŁAW, WROCŁAW, POLAND