CYCLIC VECTORS OF INDUCED REPRESENTATIONS

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In this note we prove that for a first countable locally compact group every unitary representation induced by a cyclic representation is cyclic.

This result has been recently obtained also by F. Greenleaf and M. Moskowitz [2] in a more complicated way.

Let $G$ be a first countable locally compact group. Let $\mathcal{H}$ be the space of continuous functions with compact support equipped with the Schwartz topology on $G$. Let $D$ be the cone in $\mathcal{H}'$ of positive-definite measures on $G$, i.e., $\mu \in D$ if $\langle x\star \mu, x \rangle \geq 0$ for all $x$ in $\mathcal{H}$. For each $\mu$ in $D$ we define $I = \{ x \in \mathcal{H} : \langle x\star \mu, x \rangle = 0 \}$. Then $\mathcal{H}^\mu_\mathcal{H} = \mathcal{H}/I$ is a pre-Hilbert space with a strictly positive-definite inner product $(\tilde{x}, \tilde{y})_\mu = (y\star x, \mu)$, where $x \mapsto \tilde{x}$ is the natural mapping of $\mathcal{H}$ onto $\mathcal{H}^\mu_\mathcal{H}$. Moreover, if $L_\mu x(h) = x(g^{-1}h)$, $g, h \in G$, then $I$ is stable under $L_\mu$, $g \in G$, and so $L_\mu$ acts on $\mathcal{H}^\mu_\mathcal{H}$ and is unitary with respect to $\langle \cdot, \cdot \rangle_\mu$. As such it extends to the completion $\mathcal{H}^\mu$ of $\mathcal{H}^\mu_\mathcal{H}$. Let $g \mapsto L_\mu^g$ be the representation thus obtained. If $P$ is a projection in $\mathcal{H}^\mu$ which commutes with the $L_\mu^g$, $g \in G$, then there exists a unique measure $\nu$ in $D$ such that

\[(*) \quad (P\tilde{x}, \tilde{y})_\mu = \langle y\star x, \nu \rangle \quad \text{for all } x, y \in \mathcal{H}.\]

For the details and proofs cf. [1, pp. 40–48].

Let $\{e_n\}$, $n = 1, 2, 3, \cdots$, be an approximate unit in $\mathcal{H}$ and let

$$\xi = \sum_1^\infty \lambda_n e_n^* e_n,$$

where the $\lambda_n$ are positive and such that the series is uniformly convergent.

**Theorem.** For every measure $\mu$ in $D$, the vector $\tilde{\xi}$ is cyclic with respect to $L_\mu^g$ in $\mathcal{H}^\mu$.

**Proof.** Suppose that $\tilde{\xi}$ belongs to a subspace $\mathcal{N}$ of $\mathcal{H}^\mu$ which is $L_\mu^g$ invariant, $g \in G$. Let $P$ be a projection onto $\mathcal{N}^\perp$. Clearly, $P$ commutes with $L_\mu^g$ and so it defines a measure $\nu$ in $D$ such that $(*)$ holds. Since
$P_{\xi} = 0$, for every $e_n$, $n = 1, 2, 3, \cdots$, we have $0 = (P_{\xi}, e_n) = \langle e_n^* \xi, \nu \rangle$, whence

$$\langle \xi, \nu \rangle = 0 \quad \text{and} \quad \sum_{1}^{\infty} \lambda_n (e_n^* e_n, \nu) = 0.$$  

Thus, since $\lambda_n > 0$ and $\langle e_n^* e_n, \nu \rangle \geq 0$, we have $\langle e_n^* e_n, \nu \rangle = 0$ for all $n = 1, 2, 3, \cdots$, which, by Schwartz inequality, means that $\langle e_n x, \nu \rangle = 0$ for all $x \in \mathcal{H}$ and $n = 1, 2, 3, \cdots$, so $\nu = 0$, which implies that $P = 0$ and $\mathcal{N} = \mathcal{H}_0$.

Since every unitary representation induced from a cyclic representation is of the form $g \mapsto L_g^\mu$ for a positive-definite measure $\mu$ (cf. [1]) we have

**COROLLARY.** For a first countable locally compact group every unitary representation induced by a cyclic representation is cyclic.

**REFERENCES**


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