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A REPRESENTATION THEOREM FOR L^p SPACES

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ABSTRACT. Using the theory of symmetric stable process of index $p \in (0, 2]$, we prove that if a separable Fréchet space L has all its finite dimensional subspaces linearly isometric with a subspace of $L^p[0, 1]$ then L itself is linearly isometric with a subspace of $L^p[0, 1]$.

1. **Summary.** Let p be a real number $\geqq 1$. Bretagnolle et al. [**1**] have proved that if L is a separable Banach space such that every finite dimensional subspace of L is linearly isometric with a subspace of $L^p[0, 1]$ then in fact L is itself linearly isometric with a subspace of $L^p[0, 1]$. We propose to give a simpler, more constructive proof of this fact which is in fact valid for $p \in (0, 2]$. Our techniques are an extension of the techniques in Bretagnolle et al. [**1**].

2. **Main results.** If Ω is any set and $\mathscr{A}$ is a σ-field of subsets of Ω, then for any nonnegative measure μ on $(\Omega, \mathscr{A})$ we denote, for any $p > 0$, $L^p[\Omega, \mu]$ to be the set of all real valued $\mathscr{A}$ measurable functions f on Ω such that $\int_\Omega |f(s)|^p \, d\mu(s) < \infty$. We identify functions equal a.e. For $p > 0$ we define the norm of f, denoted by $\|f\|$ to be $[\int_\Omega |f(s)|^p \, d\mu(s)]^{1/p}$. For $p \geqq 1$, the norm $\|f\|$ makes $L^p[\Omega, \mu]$ into a Banach space, while for $p \leqq 1$ the metric $\|f\|^p$ makes $L^p[\Omega, \mu]$ into a Fréchet space. If $\Omega = [0, 1]$ and μ is Lebesgue measure we simply write $L^p[0, 1]$. In the following theorem, when referring to isometries we mean the metric $\|f\|$ when $p \geqq 1$ and $\|f\|^p$ when $p < 1$. If E is a topological space and if $(\mu, \mu_1, \mu_2, \cdots)$ is a sequence of bounded measures on E, we say that μ_n converges weakly to μ if $\int_E f \, d\mu_n \to \int_E f \, d\mu$ for all bounded continuous real functions on E.

THEOREM. *Suppose L is a separable Fréchet space such that any finite dimensional subspace of L is linearly isometric with a subspace of* $L^p[0, 1]$, *where p is a fixed real number in* $(0, 2]$. *Then in fact L is itself linearly isometric with a subspace of* $L^p[0, 1]$. (*If ρ denotes the metric on L and H is a linear isometry from a subspace L_0 of L to* $L^p[0, 1]$, *then for* $p \geqq 1$ *we have* $\rho(f) = \|H(f)\|$ *for* $f \in L_0^1$, *while for* $p < 1$ *we have* $\rho(f) = \|H(f)\|^p$.

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H of course depends on the subspace L_0 but $\|H(f)\|$ does not, so we suppress this dependence from our notation.)

PROOF. For any integer $n>0$ and any choice of $f_1, \cdots, f_n \in L$, the function $\exp(-\|t_1H(f_1)+\cdots+t_nH(f_n)\|^p)$ defined for $t=(t_1, \cdots, t_n)\in R^n$ is in fact the characteristic function of a probability distribution in R^n which is symmetric stable of index p. (This is most easily seen by letting $X=(X(v)|v\in[0, 1])$ be a stochastic process such that $X(0)=0$ and such that X has time homogeneous independent increments with

$$E(\exp(i\alpha X(v))) = \exp(-|\alpha|^p v),$$

and then defining the stochastic integrals $X_m=\int_{[0,1]} H(f_m)\, dX$, for $1\leqq m\leqq n$, as in M. Schilder [4]. We then have

$$E(\exp(i(t_1X_1 + \cdots + t_nX_n))) = \exp(-\|t_1H(f_1) + \cdots + t_nH(f_n)\|^p).$$

By [3] or [1] it follows that there exists a Borel measure μ_n on the unit sphere S_n of R^n such that

$$\|t_1H(f_1) + \cdots + t_nH(f_n)\|^p = \int_{S_n} |\langle t, s\rangle|^p\, d\mu_n(s)$$

where for $s=(s_1, \cdots, s_n)\in S_n$ we let $\langle t, s\rangle=t_1s_1+\cdots+t_ns_n$. Now $1=\sum_1^n (s_m)^2$ hence $1\leqq\sum_1^n |s_m|^p$ since $p\leqq 2$. It follows that

$$\mu_n(S_n) \leqq \int_{S_n} \sum_1^n |s_m|^p\, d\mu_n(s) = \sum_1^n \|H(f_m)\|^p.$$

Let us now choose a sequence $\{f_m|m=1, \cdots, n, \cdots\}$ in L such that the closure of the linear span of $\{f_n\}$ is all of L and such that $\sum_1^\infty \|H(f_n)\|^p<\infty$.

Let E denote the set of all sequences $s=(s_1, \cdots, s_n, \cdots)$ of real numbers s_n such that $\sum_1^\infty (s_m)^2\leqq 1$. Let E have the smallest topology which makes all the functionals $s\to\langle t, s\rangle$ continuous where t varies over the set of all finite sequences $(t_1, \cdots, t_n)$ and $\langle t, s\rangle=\sum_1^n t_ms_m$ for $s\in E$; let $\mathscr{B}_E$ stand for the σ-field of subsets of E generated by the open subsets of E. Now it is well known that E is compact [2, p. 427], and it is clear that S_n is a closed subset of E. The measure μ_n defined before is now to be considered as defined on $(E, \mathscr{B}_E)$ through the map that imbeds S_n into E.

Since $\mu_n(E)=\mu_n(S_n)\leqq\sum_1^\infty \|H(f_n)\|^p$, some subsequence μ_{n_k} converges weakly to some measure μ defined on $(E, \mathscr{B}_E)$. (See [2, p. 427].)

Now for all $n_k\geqq m$ and for $t=(t_1, \cdots, t_m)$ we have

$$\int_E |\langle t, s\rangle|^p\, d\mu_{n_k}(s) = \left\| \sum_1^m t_iH(f_i) \right\|^p.$$

Hence by weak convergence we have that

$$\int_E |\langle t, s\rangle|^p\, d\mu(s) = \left\| \sum_1^m t_iH(f_i) \right\|^p.$$

So letting $e_j(s)=s_j$, it follows that the map $\sum_1^m t_i f_i \to \sum_1^m t_i e_i$ is a linear isometry from the linear span of $\{f_n\}$ into $L^p[E, \mu]$. It follows that the linear isometry can be extended to all of L. Now $L^p[E, \mu]$ is separable, hence it itself may be linearly and isometrically imbedded into $L^p[0, 1]$. Q.E.D.

REMARKS. If $p=2$ then

$$\int_E \sum_1^n (s_m)^2 \, d\mu(s) = \sum_1^n \|H(f_m)\|^2$$

and hence letting $n \to \infty$ we have that

$$\int_E \sum_1^\infty (s_m)^2 \, d\mu(s) = \sum_1^\infty \|H(f_m)\|^2.$$

Now $\mu(E) \leqq \sum_1^\infty \|H(f_m)\|^2$ hence $\sum_1^\infty (s_n)^2 = 1$ a.e. with respect to μ. In other words we conclude that μ lives on the set $S=\{s | s=(s_1, s_2, \cdots)$ with $\sum_1^\infty (s_i)^2=1\}$. For $p<2$ we can make a similar refinement to our theorem, but we leave it for a later paper.

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