ISOMETRIES OF $H^p$ SPACES OF THE TORUS

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Abstract. Denote by $H^p$ ($1 \leq p \leq \infty$) the Banach spaces of complex-valued functions in $L^p$ of the torus whose Fourier coefficients vanish off a half plane determined by a lexicographic ordering. The surjective isometries of the spaces $H^p$ ($p \neq 2$) are characterized in terms of unimodular functions on the circle and conformal maps of the disc. For $1 < p < \infty$ ($p \neq 2$) the proof depends upon a characterization of certain invariant subspaces previously given by the authors.

Let $A$ be the algebra of continuous complex-valued functions on $\{ \lambda \in C: |\lambda| = 1 \}$ which are uniform limits of polynomials in $\lambda$. Denote by $H^p(d\theta)$ the closure of $A$ in $L^p(d\theta)$ where $d\theta$ denotes normalized Lebesgue measure on the circle (norm closure for $1 \leq p < \infty$; $w^*$ closure for $p = \infty$). It is well known that the Banach spaces $H^p(d\theta)$ may be identified with the Hardy classes by associating with each function in $H^p(d\theta)$ its analytic extension to the open unit disc via the Poisson integral.

DeLeeuw, Rudin, and Wermer [1], and independently Nagasawa [6], characterized the surjective isometries of $H^1(d\theta)$ and $H^1(d\theta)$. Forelli [2] extended the characterization to $H^p(d\theta)$ for $1 < p < \infty$, $p \neq 2$. We state their results in Propositions 1 and 2.

Proposition 1. A linear operator $T$ of $H^\infty(d\theta)$ onto $H^\infty(d\theta)$ is an isometry if and only if

$$ (Tf)(\lambda) = \alpha f(\tau(\lambda)) \quad (f \in H^\infty(d\theta); |\lambda| = 1), $$

where $\alpha$ is a complex constant of modulus 1 and $\tau$ is a conformal map of the unit disc onto itself.

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Proposition 2. Let \( 1 \leq p < \infty, p \neq 2 \). A linear operator \( T \) of \( H^p(d\theta) \) onto \( H^p(d\theta) \) is an isometry if and only if

\[
(Tf)(\lambda) = \alpha(\tau(\lambda))^{1/p}f(\tau(\lambda)) \quad (f \in H^p(d\theta); |\lambda| = 1),
\]

where \( \alpha \) and \( \tau \) are as in Proposition 1.

We denote by \( A(T^2) \) the algebra of continuous, complex-valued functions on the torus \( T^2 = \{(z, w): |z| = |w| = 1\} \) which are uniform limits of polynomials in \( z^m w^n \) where \( (m, n) \in \mathcal{S} = \{(m, n): n > 0\} \cup \{(m, 0): m \geq 0\} \). Denoting normalized Haar measure on \( T^2 \) by \( dm \), we define \( H^p \) as the closure of \( A(T^2) \) in \( L^p(dm) \) (norm closure for \( 1 \leq p < \infty \); \( w^* \) closure for \( p = \infty \)). The purpose of this paper is to present characterizations of the isometries of \( H^p \) onto \( H^p \) for \( 1 \leq p \leq \infty, p \neq 2 \).

\( H^p \) consists of those functions in \( L^p(dm) \) whose double Fourier coefficients vanish off the half-plane \( \mathcal{S} \) which determines a lexicographic ordering. The maximal ideal space of \( A(T^2) \) can be identified with \( (\{z: |z| = 1\} \times \{w: |w| \leq 1\}) \cup (\{z: |z| \leq 1\} \times \{0\}) \), with \( dm \) identified with \( (z, w) = (0, 0) \). Since \( A(T^2) \) is a logmodular algebra we have at our disposal the results of [4].

We denote by \( Z \) and \( W \) the functions \( Z(z, w) = z \) and \( W(z, w) = w \). The closure in \( L^p(dm) \) of the polynomials in \( Z \) is denoted by \( Z^p \); the closure of the polynomials in \( Z^m W^n, n \geq 1 \), by \( P \); and finally the closure of the polynomials in \( Z \) and \( Z \) by \( L^p \). By [5, Lemma 5, p. 467],

\[
H^p = Z^p \oplus P
\]

for \( 1 \leq p \leq \infty \), where \( \oplus \) denotes algebraic direct sum. A function \( f \) in \( H^p \) is inner if \( |f| \equiv 1 \); \( f \) is outer if \( f \cdot A(T^2) \) is dense in \( H^p \).

Theorem 1. A linear operator \( T \) of \( H^\infty \) onto \( H^\infty \) is an isometry if and only if

\[
(Tf)(z, w) = \alpha f(\tau(z), w\sigma(z)) \quad (f \in H^\infty; |z| = |w| = 1),
\]

where \( \alpha \) is a complex constant of modulus 1, \( \tau \) is a conformal map of the unit disc onto itself, and \( \sigma \) is a unimodular measurable function.

According to [1, Theorem 3, p. 695] it suffices to prove

Theorem 2. A linear operator \( \Psi \) of \( H^\infty \) onto \( H^\infty \) is an algebra automorphism if and only if

\[
(\Psi f)(z, w) = f(\tau(z), w\sigma(z)) \quad (f \in H^\infty; |z| = |w| = 1),
\]

where \( \tau \) and \( \sigma \) are as in Theorem 1.
LEMMA 1. If $\Psi$ is an algebra automorphism of $H^\infty$, then $\Psi$ carries inner functions to inner functions, $\Psi Z^\infty = Z^\infty$, and $\Psi I^\infty = I^\infty$

PROOF. If $F$ is inner but $\Psi F$ is not, then there exists $\varepsilon > 0$ such that $m(K)>0$ where $K = \{ x : |\Psi F(x)| < 1 - \varepsilon \}$. Choose $h \in H^\infty$ with $|h(x)| = 1$ on $K$ and $|h(x)| = 1 - \varepsilon$ on $T^2 \setminus K$ [4, Theorem 5.9, p. 297]. If $\Psi g = h$, $\| Fg \|_\infty = 1$ but $\| \Psi(Fg) \|_\infty = \| (\Psi F)h \|_\infty \leq 1 - \varepsilon$. Thus $\Psi F$ is inner.

Let $M$ be the closure of $\Psi I^\infty$ in $L^2(dm)$. $M$ is clearly invariant under multiplication by functions in $H^\infty$ and also by $V$ where $V = \Psi Z$. For if $f \in I^\infty$, $fZ \in I^\infty$, so $\Psi f = \Psi(fZ)\Psi(Z)$ or $\Psi(fZ) = (\Psi f)(V)$.

If $M$ has the form $FH^2$ for some inner function $F$, then $F \cdot \bar{V} \in M$ so $\bar{V} \in H^2$. But $\Psi Z \in H^2$ so it is a constant. This contradicts the fact that $\Psi$ is injective, so $M \subseteq I^1$ [4, p. 293]. It follows, using (3), that $\Psi I^\infty \subseteq I^\infty$.

Applying the same argument to the automorphism $\Psi^{-1}$, we conclude that $\Psi^{-1}I^\infty = I^\infty$.

To show that $\Psi Z^\infty = Z^\infty$, it suffices to show that $\Psi Z \in Z^\infty$. Write $f = \Psi Z$ and suppose $f = f_1 + f_2$ where $f_1 \in Z^\infty$ and $f_2 \in I^\infty$. Then $I^\infty = \Psi(ZI^\infty) = I^\infty$, so $f_2 = fg$ for some $g \in I^\infty$. Thus $g = f_2 \bar{f} = (f - f_1) \bar{f} = 1 - f_1 \bar{f}$, which is orthogonal to $I^\infty$. Thus $g$ and hence $f_2$ vanish.

LEMMA 2. If $E_1 \in Z^\infty$ and $E_2 \in I^\infty$ are inner functions, and if for each Borel set $Y \subseteq T^2$, $\mu(Y) = m(X)$ where $X = \{(z,w) : (E_1(z), E_2(z,w)) \in Y\}$, then $\mu \ll m$.

PROOF. The Fourier-Stieltjes coefficients of $\mu$ are $\hat{\mu}(m,n) = \int E_1^m E_2^n dm$. Thus $\hat{\mu}(m,0) = a^m$ for $m \geq 0$ ($a = \int E_1 dm$). Since $E_2 \in I^\infty$, $\hat{\mu}(m,n) = 0$ for $n \geq 1$. It follows that $\mu$ is the product measure $\mu = Pdz \times dw = Qdm$, where $P(z) = (1 - |a|^2)(1 - |a z|^2)$, $dz$ and $dw$ are each Lebesgue measure, and $Q \in L^\infty(dm)$. In particular, if $Y$ is $m$-null, then $X$ is $m$-null. This argument is based on Forelli [2, p. 724].

PROOF OF THEOREM 2. By Lemma 1, $\Psi W \subseteq I^\infty$. In fact $\Psi W = W \sigma$ for some $\sigma(z) = \sigma(z,w) \in L^\infty$, as can be shown by an argument similar to that by which we showed $\Psi Z \subseteq Z^\infty$. Writing $\tau(z) = \tau(z,w) = (\Psi Z)(z,w)$, we see that $\tau$ is a conformal map of the disc by Proposition 1. Setting $E_1 = \tau$ and $E_2 = w\sigma$ in Lemma 2, we conclude that $f(\tau, w\sigma)$ is well defined for all measurable functions $f$. Thus (5) holds for all $f$ in the algebra $A$ generated by $Z^n W^n$, $(m,n) \in S$.

To establish (5) for all $f \in H^\infty$, it suffices to show that the automorphism $\Phi f = \Psi^{-1}(f(\tau, w\sigma))$ is the identity. We have seen that $\Phi Z = Z$ and $\Phi W = W$ and the proof of Proposition 1 shows that $\Phi$ is the identity on $Z^\infty$. Thus it suffices to show that $\Phi$ is the identity on $I^\infty$.

First we show that $\Phi(Z_K W) = Z_K W$ where $Z_K$ is a characteristic function in $L^\infty$. Since the function $\Phi(Z_K W)W$ is equal to its own square, it too is a characteristic function $Z_K \in L^\infty$. There remains only to show that $K = K'$.
or in fact that $K \subseteq K'$ since the argument also applies to $\Phi^{-1}$. If not, there exists a nonzero continuous function $f \in L^\infty$ with zero set $K_1 \subseteq K/K'$ of positive measure. Then

$$0 = \Phi(fW)\Phi(\chi_{K_1}W) = fW\chi_{K_1}W.$$  

(6)

Since $K_1 \subseteq K$, $K' \subseteq K'$, so $f$ does not vanish on $K_1$. This contradicts (6).

Thus $\Phi(\chi_KW) = \chi_KW$, and in general for $g \in L^\infty$, $\Phi(gW^n) = gW^n$ ($n \geq 1$). If $g \in L^\infty$, $g = \sum_{i=1}^n g_iW' + hW^n$ where $g_i \in L^\infty$ and $h \in L^\infty$. Since $\Phi g = \sum_{i=1}^n g_iW' + (\Phi h)W^n$ where $\Phi h \in L^\infty$, the Fourier coefficients of $g$ and $\Phi g$ agree, so $\Phi g = g$.

Remark. Using essentially the same argument we can show that the automorphisms of $A(T^2)$ are also given by (5) except that here $\sigma$ is continuous. However, this can more easily be done by considering the homeomorphisms of the maximal ideal space of $A(T^2)$ induced by the automorphisms of the algebra.

**Theorem 3.** A linear operator $T$ of $H^p$ onto $H^p$ ($1 \leq p < \infty$, $p \neq 2$) is an isometry if and only if

$$\Phi f(z, w) = \Phi \left( \alpha(T) f(z) \right) \frac{1}{\|f\|_p} f(z, w),$$

(7)

for all $f \in H^p$, where $|z| = |w| = 1$, $\alpha$ is a complex constant of modulus 1, $\tau$ is a conformal map of the unit disc onto itself, and $\sigma$ is a unimodular measurable function on the circle.

The proof depends on our results in [5] on the characterization of sesqui-invariant subspaces of $H^p$. A closed subspace $M \subseteq H^p$ is called invariant if $fM \subseteq M$ for all $f \in H^\infty$. An invariant subspace $M \subseteq H^p$ is called sesqui-invariant if $ZM \subseteq M$ and simply invariant if this is not the case. If $M$ is sesqui-invariant, it has the form

$$M = \chi_E \cdot \psi \cdot \Pi^p$$

where $\psi$ is unimodular and $\chi_E$ is the characteristic function of the support set of $M$ ([5, Theorem 3, p. 471]; see also [5, p. 473 for the torus case]). If $M$ is simply invariant, it has the form $M = \psi H^p$ ($\psi$ unimodular) by the generalized Beurling theorem [8].

**Lemma 3.** Let $F = T(1)$ and $E$ be the support set of $F$. Then $m(E) = 1$.

**Proof.** Since $F \in H^p$, $\chi_E$ is independent of $w$, so $G = w(1 - \chi_E) \in H^p$. Let $g = T^{-1}(G)$. Thus

$$\int |1 \pm g|^p \, dm = \int |F \pm G|^p \, dm$$

$$= \int |F|^p \, dm + \int |G|^p \, dm = 1 + \int |g|^p \, dm.$$
Therefore
\[ \int |1 + g|^p \, dm + \int |1 - g|^p \, dm = 2 \left[ 1 + \int |g|^p \, dm \right]. \]

By [7, p. 275], \( g = 0 \) a.e., so \( m(E) = 1 \).

**Proof of Theorem 3.** Lemma 3 insures that \( dv = |F|^p \, dm \) and \( dm \) are mutually absolutely continuous. Using Forelli’s argument [2, Proposition 2, p. 723] it follows that \( Sf = Tf/F \) defines an isometry \( S \) of \( H^p \) into \( L^p(dv) \) which takes the algebra \( \mathcal{A} \) generated by \( Z^m W^n, (m, n) \in \mathcal{I} \), multiplicatively into \( L^\infty(dv) \). Write \( E_1 = S(Z) \) and \( E_2 = S(W) \). For \( f \in \mathcal{A} \), we have

\[ (8) \quad Tf(z, w) = F \cdot f(E_1, E_2). \]

We show that \( E_1 \in Z^\infty \) and \( E_2 \in L^\infty \). Since \( F \in H^p \), the sesqui-invariant subspace generated by \( F \) has the form \( JF \), where \( J \) is unimodular. Thus \( F = JG \) where \( G \in P \) and the sesqui-invariant subspace generated by \( G \) is \( P \). For \( f \in S(\mathcal{A}) \), \( WF \in F \), and the property of \( G \) insures that \( W^2 Jf \in F \). Thus the invariant subspace generated by \( S(\mathcal{A}) \) has the form \( \psi P \) or \( \psi H^p \), \( \psi \) unimodular.

In the first case \( f \psi W \in \psi P \), so \( f \in L^\infty \) for all \( f \in S(\mathcal{A}) \) and similarly for the second case. In particular \( E_1 \in L^\infty \) and \( E_2 \in L^\infty \). The same argument applied to the algebra generated by \( Z^n W^m, n \geq 1 \), shows that \( E_1 \in L^\infty \) and \( E_2 \in L^\infty \).

We conclude that \( F \notin P \) (otherwise \( T \) would map \( H^p \) onto \( P \)). Thus \( \int \log |F| \, dm > -\infty \), so \( F = JG \) where now \( J \) is inner and \( G \) is outer. Also \( \{ fE_1^m \}, m \geq 0 \), generate a simply invariant subspace, so by the usual argument \( E_1 \in Z^\infty \). Since \( G \) is outer, the invariant subspace \( N \) generated by \( \{ JE_1^m E_2^n \}, n > 0 \), is contained in \( H^p \). Since \( N = \psi H^p \) would imply that \( E_1 \in H^p \), we have \( N \subseteq P \) so \( Je_1 \in P \), \( J \in H^p \) but \( J \notin P \) (because \( F \notin P \)), so \( E_2 \in L^\infty \). Thus \( T \) takes \( P \) into \( P \).

Thus using Lemma 2, \( f(E_1, E_2) \) is well defined for all measurable functions \( f \). The density of \( \mathcal{A} \) in \( H^p, 1 \leq p < \infty \), insures that \( (8) \) holds for all \( f \in H^p \). Imitating Forelli’s argument [2, p. 726] one shows that the function \( Q \) constructed in the proof of Lemma 2 satisfies

\[ (9) \quad \int_X |F|^p \, dm = \int_X 1/Q(E_1) \, dm \]

for all Borel sets \( X \subseteq T^3 \). Since \( T \) is surjective, both \( T \) and \( T^{-1} \) carry \( P \) into \( P \), so that \( TP = P \). Again using the argument of [2] (beginning at the bottom of p. 726) it follows that \( E_1 \), considered as a function of \( z \) alone, is a.e. the boundary value function of a conformal map \( \tau_1 \) of the disc onto itself. Define \( \tau(z, w) = \tau_1(z) \). We have \( |\tau'| = 1/Q(\tau) \) and \( (9) \) becomes
\[ \int_X |F|^p \, dm = \int_X |\tau'| \, dm \]

for all Borel sets \( X \subseteq T^2 \). Thus \( F \) and \((\tau')^{1/p}\) have the same modulus. Since the latter is outer, we can show that they differ by a constant of modulus one by showing \( F \) is outer. If \( F = JG \), \( J \) inner and \( G \) outer, then \( GH^p = H^p = TH^p = FH^p \). Dividing by \( G \), \( H^p = JH^p \), so \( J \equiv \alpha \), \(|\alpha| = 1\).

To complete the proof it suffices to show that \( E_\sigma = W_\sigma \) where \( \sigma \in L^\infty \).

For this we need to show that \( \sigma I^p = I^p \) (see the analogous argument for \( p = \infty \)). But since \( F \) and \( 1/F \) are bounded, \( \sigma I^p = \overline{W(SW)(SI^p)} = \overline{WS(WI^p)} = \overline{WWI^p} = I^p \).

For the case \( p = 1 \), Theorem 3 can also be obtained by adapting the original argument of deLeeuw, Rudin and Wermer [1, Theorem 2, p. 694] in which they deduce the isometries of \( H^1(d\theta) \) by exploiting the special properties of the extreme points of the unit ball of \( H^1(d\theta) \). To do this one needs three facts about functions on the torus: (a) the extreme points of the unit ball of \( H^1 \) are the outer functions of norm one (Gamelin [3]), (b) the identity \( \int f \, dm = \int (\tau, w) \tau' \, dm \) (a straightforward calculation), and (c) the result of Lemma 4 below. Let \( B^* \) be the set of extreme points in the unit ball of \( H^1 \), \( P(m) = \{z: |z| < 1\} \times \{0\} \), and \( D_z = \{z\} \times \{w: |w| < 1\} \) for each \( |z| = 1 \).

LEMMA 4. A function \( f \in H^1 \) of norm 1 lies in the closure of \( B^* \) if and only if

(10) \( f \) has no zeros on \( P(m) \) and \( f \) has no zeros on \( D_z \) for almost all \( z \).

PROOF. If \( f \) lies in the closure of \( B^* \), then there exist \( f_n \in B^* \) converging uniformly on compact sets to \( f \) on \( P(m) \) and on each \( D_z \) for almost all \( z \).

(10) follows.

Conversely suppose (10) holds. Define \( f_r(z, w) = f(\frac{z}{r}, rw) \), \( 0 < r < 1 \). Let \( f_r = F_r g_r, F_r \) inner, \( g_r \) outer. One shows that \( F_r \) is independent of \( r \), say \( F_r = F \in \mathbb{C}^* \). Let \( h_r(z, w) = F(rz) \). Then \( f \) is the \( L^1 \) limit of the outer functions \( h_r g_r \), so \( f \) lies in the closure of \( B^* \).

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