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# STONE'S THEOREM FOR A GROUP OF UNITARY OPERATORS OVER A HILBERT SPACE

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ABSTRACT. The spectral representation for a group of unitary operators acting on a Hilbert space where the parameter set is a separable real Hilbert space is obtained. The usual spectral representation of such a group of unitary operators is when the parameter set is a locally compact abelian group (Stone's theorem). The main result used in the proof is the Bochner theorem on the representation of positive definite functions on a real Hilbert space.

**Introduction.** This paper discusses a spectral representation theorem for a group of unitary operators acting on a Hilbert space $H$, where the parameter set is a real Hilbert space $T$. The usual spectral representation of such a group of unitary operators (the well-known Stone theorem) is when the set $T$ is a locally compact abelian group [**3**, p. 392]. The main result used in the course of our work is the Bochner theorem on the representation of positive definite functions on a real Hilbert space $T$. This theorem is included in the interesting work of L. Gross [**2**] in *Harmonic analysis on Hilbert space*.

1. **Preliminaries.** We begin this section by introducing some notation which will be used later. $T$ will be a real Hilbert space. We will use $s$, $t$, etc. for points in $T$. By a Borel set in $T$ we shall mean a set in the $\sigma$-algebra determined by the collection of open sets in $T$. In $T$ the inner product and norm will be denoted by $(\cdot\,,\cdot)_T$ and $|\cdot|_T$. A complex-valued function $\varphi$ on $T$ will be called positive definite if for every finite set $t_1, \cdots, t_n$ of points in $T$ the matrix $(a_{ij})$ defined by $a_{ij}=\varphi(t_i-t_j)$ is nonnegative definite. The following definition is due to L. Gross [**2**, p. 5].

1.1 DEFINITION (TOPOLOGY $\tau$ FOR $T$). The topology of $\tau$ is defined as the weakest topology for which all Hilbert-Schmidt operators on $T$ into $T$ are continuous from $\tau$ to the strong topology of $T$.

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In these terms we can state Bochner's theorem as proved by L. Gross [**2**, p. 20], as follows:

1.2 THEOREM. *A complex-valued function $\varphi$ on $T$ is positive definite and continuous with respect to the $\tau$-topology if and only if*

$$\varphi(t) = \int_T e^{i(t,s)}\mu(ds), \tag{1.3}$$

*where $\mu$ is a positive finite measure on $T$.*

It is known that if $T$ is separable then $\varphi$ determines $\mu$ uniquely (cf. [**2**, p. 1 and p. 6]). Since a complex measure $\mu$ can be decomposed into $\mu=\mu_1-\mu_2+i(\mu_3-\mu_4)$, where $\mu_1$, $\mu_2$, $\mu_3$, $\mu_4$ are nonnegative finite measures, we can state the following corollary:

1.4 COROLLARY. *If $\psi$ is any complex-valued function defined on a separable real Hilbert space $T$ which has the form*

$$\psi(t) = \int_T e^{i(t,s)}\, d\mu,$$

*when $\mu$ is a complex-valued measure, then $\mu$ is uniquely determined by $\psi$.*

The groups of unitary operators whose spectral representation we shall establish will be denoted by $\{U_t, t\in T\}$. Precisely speaking we have a complex Hilbert space $H$ {with inner product $(\cdot\,,\cdot)_H$ and norm $|\cdot|_H$} and for each $t\in T$, we have a unitary operator $U_t$ acting on $H$ such that $U_tU_s=U_{t+s}$ for all $s$, $t\in T$. Points of $H$ will be denoted by $x$, $y$, etc. We shall assume in proving our spectral theorem that $U_t$ is weakly $\tau$-continuous in $t$, i.e., $(U_tx, y)$ is continuous with respect to the $\tau$-topology for each $x, y\in H$. We remark that our proof of Theorem 2 uses the representation of positive definite $\tau$-continuous functions $\varphi$ [**1**, p. 20], in the same spirit as the corresponding Bochner theorem concerning positive definite functions on a locally compact abelian group was used to obtain the Stone's theorem for a group of unitary operators over a locally compact abelian group.

2. **The spectral theorem.** By a spectral measure on $T$ for a given Hilbert space $H$ we shall mean a family of operators $\{E(B)\}$ on $H$ into $H$ on $\mathscr{B}(T)$, the Borel sets in $T$ with the following properties:

(2.1)
- (a) $E(B)$ is an orthogonal projection operator.
- (b) $E(\varnothing)=0$.
- (c) $E(T)=I$.
- (d) If $B=\bigcup_i B_i$, when $B_i$ are disjoint sets in $\mathscr{B}(T)$, then $E(B)=\sum_i E(B_i)$.

We can now state and prove our spectral theorem.

2.2 THEOREM. *Let $\{U_t, t\in T\}$ be a weakly $\tau$-continuous group of unitary operators on a given Hilbert space $H$, over a separable Hilbert space $T$. Then there exists a unique spectral measure $\{E(B), B\in\mathscr{B}(T)\}$ such that*

$$U_t = \int_T e^{i(t,s)}\, dE(s), \qquad t \in T. \tag{2.3}$$

*A bounded operator commutes with every $U_t$ if and only if it commutes with each $E(B)$, $B\in\mathscr{B}(T)$.*

PROOF. For each $x\in H$, the function $(U_t x, x)$ is a positive definite function. Since by assumption $(U_t x, x)$ is $\tau$-continuous, by Theorem 1.2 and (1.3),

$$(U_t x, x) = \int_T e^{i(t,s)} \mu_x(ds). \tag{2.4}$$

From (2.4) it follows that for each $x, y\in H$ there exists a complex-valued measure $\mu_{x,y}$ such that

$$(U_t x, y) = \int_T e^{i(t,s)} \mu_{x,y}(ds). \tag{2.5}$$

The uniqueness of this representation (Corollary 1.4) then shows that for each $B\in\mathscr{B}(T)$, $\mu_{x,y}(B)$ is a bilinear functional and also

$$\mu_{x,y}(B) = \overline{\mu_{y,x}(B)}. \tag{2.6}$$

We note that $|\mu_{x,y}(B)|^2 \leqq \mu_{x,x}(B)\mu_{y,y}(B)$. Putting $t=0$ in (2.4) we get that

$$\mu_{x,x}(B) \leqq |x|^2; \tag{2.7}$$

then (2.6) and (2.7) imply that $\mu_{x,y}(B)$ is a bounded bilinear functional on $H$. Hence by [3, p. 202], we conclude that for each Borel set $B\in\mathscr{B}(T)$ there is a bounded operator $E(B)$ on $H$ into $H$ such that $(E(B)x, y) = \mu_{x,y}(B)$, $E(B) = E^*(B)$. If $B_1, B_2\in\mathscr{B}(T)$ we have

$$\begin{aligned}(U_{t+s}x, y) &= \int_T e^{i(t+s,\xi)}(E(d\xi)x, y)\\ &= \int_T e^{i(t,\xi)} e^{i(t,\xi)}(E(d\xi)x, y) = \int e^{i(t,\xi)} \nu(d\xi),\end{aligned}$$

when $\nu(d\xi) = e^{i(s,\xi)}(E(d\xi)x, y)$. We also have

$$(U_{t+s}x, y) = \int e^{i(t,\xi)}(E(d\xi)U_s x, y).$$

By the uniqueness result (Corollary 1.4), we have that, for each $B_1\in\mathscr{B}(T)$,

$$\int e^{i(s,\xi)}(E((d\xi)\cap B_1)x, y) = \int_{B_1} e^{i(s,\xi)}(E(d\xi)x, y) = (E(B_1)U_s x, y). \tag{2.8}$$

Also we have

$$(2.9) \qquad (E(B_1)U_s x, y) = (U_s x, E(B_1)y) = \int e^{i(s,\xi)}(E(d\xi)x, E(B_1)y).$$

From (2.8) and (2.9) we obtain $(E(B\cap B_1)x, Y)=(E(B)x, E(B_1)Y)$. Therefore $E(B\cap B_1)=E(B)E(B_1)$. Since

$$(U_0x, y) = (x, y) = \int \mu_{x,y}(ds) = \mu_{x,y}(T)$$

we get that, for all $x, y\in H$, $(E(T)x, y)=(x, y)$; hence $E(T)=I$. Let $B=\bigcup B_i$, $B_i$'s disjoint in $\mathscr{B}(T)$. Then

$$(E(B)x, y) = \mu_{x,y}(B) = \mu_{x,y}(\bigcup B_i) = \sum_i \mu_{x,y}(B_i) = \sum_i (E(B_i)x, y).$$

Hence, $E(B)=\sum_i E(B_i)$.

This implies $E(\varnothing)=2E(\varnothing)$ and hence $E(\varnothing)=0$. So that $E$ is a spectral measure. The existence of the integral $\int_T e^{i(t,\xi)}E(d\xi)$ which we call $V_t$ is well known. Then from the usual operational calculus it follows that $(V_tx, y)=\int e^{i(t,\xi)}(dE(w)x, y)$, so that if each $t\in T$ we have $(V_tx, y)=(U_tx, y)$, and hence $V_t=U_t$.

In view of Corollary 1.4 the uniqueness of $E(\cdot)$ is clear.

Now let $V$ be any bounded linear operator on $H$ into $H$. An argument similar to the one given in [**1**, p. 593] shows that $V$ commutes with $U_t$, $t\in T$, if and only if $V$ commutes with $E(B)$, $B\in\mathscr{B}(T)$.

3. **Application to stochastic processes.** Let $(\Omega, B, P)$ be a probability space and let $\{X_t\}$ be a 2nd order stochastic process with $t\in T$ $\{T$ is a separable real Hilbert space$\}$. $\mathscr{E}$ will denote the expectation operator. If $\mathscr{E}(X_t\bar{X}_s)$ depends only on the difference of $t-s$ then $\{X_t\}$ is said to be a stationary stochastic process. The theory of such processes over any locally compact abelian group has been studied [**4**]. The same line of proof as in [**4**] can be used to prove:

3.1 THEOREM. (i) *Let $\{X_t, t\in T\}$ be a stationary stochastic process with values in $L_2(\Omega, B, P)$.*

(ii) *Let $H=$closed subspace in $L_2(\Omega, B, P)$ spanned by $\{X_t, t\in T\}$.*

(iii) *Let $\{X_t\}$ be $\tau$-continuous, i.e. the correlation function $\mathscr{E}(X_t\bar{X}_0)$ is continuous with respect to the $\tau$-topology of T.*

*Then*

(a) *There exists a unique spectral measure E on $\mathscr{B}(T)$ for the Hilbert space H such that*

$$(3.2) \quad X_t = \int_T e^{i(t,\xi)}E(d\xi)X_0, \qquad \mathscr{E}(X_t\bar{X}_s) = \int_T e^{i(t-s,\xi)}(E(d\xi)X_0, X_0).$$

*The measure* $dF=(E(d\xi)X_0, X_0)$ *is called the spectral distribution of the process* $\{X_t, t\in T\}$.

(b) *The mapping* $e^{i(t,\cdot)}\to X_t$ *is an isometry on* $L_{2,F}$ *onto* $H$.

Proof of (b) depends on the fact that $e^{i(t,\cdot)}$ are dense in $L_{2,F}(T)$ because of Corollary 1.4.

REMARK. Theorem 3.1 and representation (3.2) can be used in the prediction theory of stationary stochastic processes over a separable Hilbert space to obtain results similar to the usual work in the prediction of stationary stochastic processes over a locally compact abelian group [**4**]. This work is being investigated by the author and the result will be published elsewhere.

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