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CHARACTERIZATION OF RINGS USING QUASIPROJECTIVE MODULES. III

JONATHAN S. GOLAN

ABSTRACT. A ring R is regular [completely reducible] if and only if the character module of every left R-module is quasi-injective [quasiprojective]. Submodules of quasiprojective left R-modules over a left perfect ring R are quasiprojective if and only if singular left R-modules are injective. A splitting theorem for right perfect rings over which submodules of quasiprojective left R-modules are quasiprojective is also proven. These results continue the author's previous work ([5] and [6]).

A left R-module M is *quasiprojective* if and only if, for every epimorphism $\alpha: {}_RM \to {}_RN$, $\mathrm{Hom}_R(M, N) = \mathrm{Hom}_R(M, M)\alpha$. Dually, M is *quasi-injective* if and only if, for every monomorphism $\beta: {}_RN \to {}_RM$, $\mathrm{Hom}_R(N, M) = \beta\, \mathrm{Hom}_R(M, M)$. In [5] and [6] we showed how quasi-projective modules can be used to characterize the rings over which they are defined. Here we will prove additional ring characterizations using quasiprojective and quasi-injective modules.

In what follows R will always denote an associative ring with identity and modules and morphisms will always be taken from the category R-mod [resp. mod-R] of left [resp. right] unitary R-modules. Morphisms will always be written acting on the side opposite ring multiplication.

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1. **Character modules.** Let G be an injective cogenerator over the ring Z of integers and let $\chi = \mathrm{Hom}_Z(\ ,G)$ be the G-character functor. Then χ can be considered as a faithful exact contravariant functor R-mod$\to$mod-R [or mod-$R\to R$-mod] which commutes with finite direct sums. Furthermore, for any R-module M we have a canonical

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embedding $M \to \chi^2(M)$. The module $\chi(M)$ is called the *character module of* M (with respect to G). The major results of this section are:

THEOREM A. *A ring R is regular (in the sense of von Neumann) if and only if the character module of every left R-module is quasi-injective.*

THEOREM B. *The following are equivalent for a ring R:*
(1) *R is completely reducible.*
(2) *$\chi(M)$ is quasiprojective for every right [left] R-module M.*
(3) *$\chi^2(M)$ is quasiprojective for every left [right] R-module M.*

We will adopt the following notation from [**11**]: If M is a left R-module, then

$\pi^{-1}(M)$ will denote the class of all epimorphisms $\alpha: {}_RU \to {}_RV$ such that $\mathrm{Hom}\,(\mathrm{id}_M, \alpha): \mathrm{Hom}_R\,(M, U) \to \mathrm{Hom}_R\,(M, V)$ is an epimorphism.

$\iota^{-1}(M)$ will denote the class of all monomorphisms $\beta: {}_RU \to {}_RV$ such that $\mathrm{Hom}\,(\beta, \mathrm{id}_M): \mathrm{Hom}_R\,(V, M) \to \mathrm{Hom}_R\,(U, M)$ is an epimorphism.

$\tau^{-1}(M)$ will denote the class of all monomorphisms $\gamma: U_R \to V_R$ such that $\gamma \otimes \mathrm{id}_M: U \otimes_R M \to V \otimes_R M$ is a monomorphism.

Let M be a left R-module and U a right R-module. As a special case of the homological duality formulae [**3**, Propositions II,5.2 and II,5.2′] we have the canonical isomorphisms

$$\mathrm{Hom}_R\,(U, \chi(M)) \cong \chi(U \otimes_R M) \cong \mathrm{Hom}_R\,(M, \chi(U))$$

from which we can immediately infer:

(1.1) LEMMA. *If M is a left R-module and $\alpha: U_R \to V_R$ is a homomorphism, then the following are equivalent:*
(i) $\alpha \in \tau^{-1}(M)$.
(ii) $\alpha \in \iota^{-1}(\chi(M))$.
(iii) $\chi(\alpha) \in \pi^{-1}(M)$.

(See [**11**, Proposition 3.2].)

If M is a finitely-presented left R-module and N is an arbitrary left R-module, then as a special case of the isomorphism given in [**1**, p. 63, Exercise 14] we have a canonical isomorphism

$$\chi(N) \otimes_R M \cong \chi(\mathrm{Hom}_R(M, N)).$$

From this and Lemma 1.1 we then have:

(1.2) LEMMA. *Let M be a finitely-presented left R-module and let $\beta: {}_RU \to {}_RV$ be an epimorphism. Then the following are equivalent:*
(i) $\beta \in \pi^{-1}(M)$.
(ii) $\chi(\beta) \in \iota^{-1}(\chi(M))$.
(iii) $\chi^2(\beta) \in \pi^{-1}(M)$.

(1.3) COROLLARY. *If M is a finitely-presented left R-module and $\chi(M)$ is quasi-injective then M is quasiprojective.*

PROOF. If $\alpha: {}_RM \to {}_RV$ is an epimorphism then $\chi(\alpha): \chi(V)_R \to \chi(M)_R$ is a monomorphism and so, by the quasi-injectivity of $\chi(M)$, belongs to $\iota^{-1}(\chi(M))$. By Lemma 1.2, α then belongs to $\pi^{-1}(M)$, proving M quasi-projective. □

We are now in a position to prove our main theorems.

PROOF OF THEOREM A. If R is regular then every left R-module is flat. Lambek [**9**, p. 131] has shown that a left R-module M is flat if and only if $\chi(M)$ is injective so, in particular, $\chi(M)$ is clearly quasi-injective.

Conversely assume that R is not regular. Then there exists a left R-module M which is not flat; that is to say, there exist right R-modules U_R and V_R together with a monomorphism $\lambda: U_R \to V_R$ such that $\lambda \otimes \mathrm{id}_M: U \otimes_R M \to V \otimes_R M$ is not a monomorphism. Let ${}_RT = \chi(V) \oplus M$. Then $\chi(T) \cong \chi^2(V) \oplus \chi(M)$ and we have the canonical embeddings $\rho: V \to \chi^2(V) \to \chi^2(V) \oplus \chi(M) \cong \chi(T)$ and $\sigma: M \to T$.

If $0 \neq \sum_{i=1}^n (u_i \otimes m_i) \in \ker(\lambda \otimes \mathrm{id}_M)$ then $0 \neq \sum_{i=1}^n (u_i \otimes m_i \sigma) \in U \otimes_R T$ and

$$\left[\sum_{i=1}^n (u_i \otimes m_i \sigma)\right](\lambda\rho \otimes \mathrm{id}_T) = \left[\sum_{i=1}^n (u_i \otimes m_i)\right](\lambda \otimes \mathrm{id}_M)(\rho \otimes \sigma) = 0,$$

proving that $\lambda\rho \otimes \mathrm{id}_T: U \otimes_R T \to \chi(T) \otimes_R T$ is not a monomorphism, i.e., $\lambda\rho \notin \tau^{-1}(T)$. By Lemma 1.1, this means that $\lambda\rho \notin \iota^{-1}(\chi(T))$, proving that $\chi(T)$ is not quasi-injective. □

PROOF OF THEOREM B. The implications (1)⇒(2)⇒(3) are trivial. Therefore assume (3). If N is an injective left R-module then N can be canonically embedded in $\chi^2(N)$ and so is a direct summand of $\chi^2(N)$. Since $\chi^2(N)$ is quasiprojective, so is N and so we have that every injective left R-module is quasiprojective, which proves that R is a quasifrobenius ring [**2**, Corollary 2.3]. In particular, R is left noetherian.

Every finitely-generated left R-module over a left noetherian ring is finitely-presented [**1**, p. 36]. Since R is completely reducible iff every finitely-generated left R-module is quasiprojective (the proof is the same as that of Theorem 1.3 of [**5**]) it suffices by Corollary 1.3 to show that $\chi(M)$ is quasi-injective for every finitely-generated left R-module M. Let $\alpha: U_R \to \chi(M)_R$ be a monomorphism. Then $\chi^3(\alpha): \chi^4(M) \to \chi^3(U)$ is an epimorphism. But $\chi^4(M) \oplus \chi^3(U) \cong \chi^2(\chi^2(M) \oplus \chi(U))$ which is quasi-projective. Since a sufficient condition for an epimorphism ${}_RX \to {}_RY$ to be a retraction is that $X \oplus Y$ be quasiprojective [**6**, Lemma 2.1], it follows that $\chi^3(\alpha)$ is a retraction and hence surely belongs to $\pi^{-1}(M)$. By Lemma

1.2, $\chi(\alpha)$ therefore belongs to $\pi^{-1}(M)$ and so Lemma 1.1 implies that α belongs to $\iota^{-1}(\chi(M))$. This proves that $\chi(M)$ is quasi-injective. $\square$

A submodule N of a left R-module M is *pure* if and only if, for every right R-module T, the canonical imbedding $N \to M$ belongs to $\tau^{-1}(T)$. An R-module U is *pure-injective* if and only if, for every pure submodule N of an R-module M, the embedding $N \to M$ belongs to $\iota^{-1}(U)$.

If as our injective cogenerator G we take as a special case $\boldsymbol{R}/\boldsymbol{Z}$, where $\boldsymbol{R}$ is the field of reals, then $\chi^2(M)$ is a compact R-module and M is a pure submodule of $\chi^2(M)$. Moreover, a module U is pure-injective if and only if it is a summand of $\chi^2(U)$ [**12**, pp. 706–708]. We then get the following corollary to Theorem B, generalizing a result of Griffith [**7**]:

(1.4) COROLLARY. *A ring R is completely reducible if and only if every pure-injective left R-module is quasiprojective.*

2. **Heredity with respect to quasiprojectives.** In [**6**] we looked briefly at rings R satisfying the condition that every [finitely-generated] submodule of a quasiprojective left R-module be quasiprojective. Such rings we will call *left* [*semi-*] *HQ-rings*. This condition is stronger than left hereditary and in fact we have shown [**6**, Theorem 5.1] that every factor ring of a left [semi-] HQ-ring is left [semi-] hereditary. If R is left perfect then the converse also holds. We will now characterize these rings in more detail; our main results are:

THEOREM C. *A left perfect ring R is a left HQ-ring if and only if every singular left R-module is injective.*

THEOREM D. *If R is a right perfect left semi-HQ ring then R admits a splitting $R = S \oplus J(R)$ over $\boldsymbol{Z}$, where S is a completely reducible subring of R containing* 1.

A two-sided ideal I of a ring R is *left T-nilpotent* if and only if, for every sequence $\langle a_i \rangle$ of elements of I, there exists an integer n such that $a_1 \cdot \ldots \cdot a_n = 0$. Ideals satisfying this condition were first studied by Levitzki [**10**]. The following lemma is known:

(2.1) LEMMA. *Let I be a left T-nilpotent two-sided ideal of a ring R. Then for every left R-module M, IM is small in M.*

PROOF. ${}_RU$ is small in ${}_RV$ if and only if $U + {}_RW = V$ implies $W = V$. Assume IM is not small in M. We will arrive at a contradiction by constructing a sequence $\langle a_i \rangle$ of elements of I such that $a_1 \cdot \ldots \cdot a_n \neq 0$ for all n.

Since IM is not small in M there exists a proper submodule N of M with $IM + N = M$. In particular there exists $a_1 \in I$ and $m_1 \in M$ such

that $a_1m_1 \notin N$ (and hence surely $a_1m_1 \neq 0$). Now assume inductively that there exists $a_1, \cdots, a_n \in I$ and $m_n \in M$ such that $a_1 \cdot \ldots \cdot a_n m_n \notin N$. Then $m_n = y + \sum_{j=1}^{k} b_j x_j$ for $y \in N$, $b_j \in I$, and $x_j \in M$. Since $a_1 \cdot \ldots \cdot a_n m_n \notin N$ there exists some j such that $a_1 \cdot \ldots \cdot a_n b_j x_j \notin N$. Take $a_{n+1} = b_j$ and $m_{n+1} = x_j$. □

(2.2) PROPOSITION. *Let R be a left HQ-ring and let H be a left T-nilpotent two-sided ideal of R. Then $H^2 = 0$. In particular, if R is left perfect then $J(R)^2 = 0$.*

PROOF. Since H is left T-nilpotent, $H = HR$ is small in R and so H is contained in the Jacobson radical $J(R)$ of R (which equals the union of all small left ideals of R). Let $S = R/H^2$. Since R is a left HQ-ring, R/H^2 is left hereditary and so, in particular, $J(R)/H^2$ is a projective left S-module since it is a left ideal of S. Let ${}_SU = J(R)/H^3$ and ${}_SV = H^2/H^3$. Then U/V is S-isomorphic to $J(R)/H^2$ and so is projective over S. But $V \subseteq HU$ and so is small in U. Thus by the splitting of the exact sequence $0 \to V \to U \to U/V \to 0$ over S we see that V is a small direct summand of U and so $V = 0$. This proves that $H^2 = H^3$. But by Lemma 2.1, H^3 is small in H^2 and so $H^2 = 0$.

The second part of the theorem comes from the well-known fact that the Jacobson radical of a left perfect ring is left T-nilpotent. □

(2.3) COROLLARY. *Let R be a left hereditary left perfect ring. Then the following are equivalent*:

(1) *R is a left HQ-ring.*

(2) $J(R)^2 = 0$.

PROOF. (1)$\Rightarrow$(2) follows directly from Proposition 2.2.

(2)$\Rightarrow$(1): Harada has shown that if R is left hereditary and semiprimary and if k is the index of nilpotency of $J(R)$ then for any two-sided ideal I of R, l.gl. dim $(R/I) \leqq k-1$ [**8**, Theorem 6]. In particular, (2) implies that R/I is left hereditary for every two-sided ideal I of R and so R is a left HQ-ring since it is left perfect. □

Recall that a submodule N of a module M is *large* if and only if it has a nonzero intersection with any nonzero submodule of M and that the singular submodule $Z(M)$ of a left R-module M is given by $Z(M) = \{m \in M \mid (0:m)$ is a large left ideal of $R\}$. An R-module M is *singular* if and only if $Z(M) = M$ and *nonsingular* if and only if $Z(M) = 0$. Both the class of singular R-modules and the class of nonsingular R-modules are closed under taking submodules.

PROOF OF THEOREM C. Let R be a left perfect ring and assume that R is a left HQ-ring as well. Let M be a singular left R-module. To show that

M is injective it suffices to show that every diagram of the form

$$\begin{tikzcd} 0 \arrow[r, dashed] & {}_RH \arrow[r, dashed, "\lambda"] \arrow[d, dashed, "\alpha"'] & R \arrow[dl, dashed, "\beta"] \\ & M & \end{tikzcd}$$

can be completed commutatively, where H is a large left ideal of R, λ is the canonical embedding, and α is an arbitrary nonzero homomorphism [**9**, p. 93].

Let $K=\ker(\alpha)$ and suppose that K is not large in R. Then there exists a nonzero left ideal I of R such that $K\cap I=0$. Since H is large in R, $I'=H\cap I$ is nonzero and $I'\cap K=0$. Thus the restriction of α to I' is a monomorphism and so $I'\cong I'\alpha$ which is a submodule of a singular module and so is singular. On the other hand, R is in particular left hereditary and so is nonsingular. Therefore I', a submodule of R, is also nonsingular, yielding a contradiction which proves that K is large in R.

By Proposition 2.2, $J(R)^2=0$ and so $J(R)$ is a left $R/J(R)$-module and so is completely reducible for, by the left perfectness of R, $R/J(R)$ is a completely reducible ring. Therefore $J(R)\subseteq \operatorname{soc}({}_RR)$. But the socle of a left R-module is precisely the intersection of all of its large submodules and so, in particular, $J(R)\subseteq K$. Therefore R/K is a left $R/J(R)$-module and thus is completely reducible.

The map α induces a monomorphism $\bar{\alpha}:H/K\to M$. Since R/K is a completely reducible R-module, H/K is a direct summand of R/K and so we have a canonical projection $\pi:R/K\to H/K$. Finally, let $\nu:R\to R/K$ be the canonical epimorphism. Then $\nu\pi\bar{\alpha}$ is an R-homomorphism $R\to M$ and for every $h\in H$, $h\nu\pi\bar{\alpha}=(h+K)\pi\bar{\alpha}=(h+K)\bar{\alpha}=h\alpha$ so $\beta=\nu\pi\bar{\alpha}$ is the homomorphism we seek.

Conversely, assume that R is a left perfect ring over which every singular module is injective. Let I be a two-sided ideal of R and let $S=R/I$. For M a left S-module let ${}_S\hat{M}$ be the injective hull of M over S. Then ${}_RM$ is large in ${}_R\hat{M}$ and so, as is easily seen, $\hat{M}/M$ is a singular left R-module. Therefore $\hat{M}/M$ is injective over R and hence injective over S. Since the sequence $0\to M\to\hat{M}\to\hat{M}/M\to 0$ is exact over S, $\operatorname{inj\,dim}({}_SM)\leqq 1$. Since this is true for any left S-module M, S is left hereditary [**3**, p. 112] and so R is a left HQ-ring by Theorem 5.1 of [**6**]. □

Note that the argument in the above proof is based on the techniques of [**4**].

(2.4) COROLLARY. *If R is a left perfect left QH-ring then $Z(M)$ is a direct summand of M for every left R-module M.*

We now turn to look at right perfect left semi-HQ-rings, and use techniques based on those of Zaks [13] to obtain a classical splitting theorem (Theorem D).

(2.5) LEMMA. *Let R be a right perfect left semi-HQ-ring. If e and f are indecomposable idempotents of R with eRf and fRe nonzero, then $Re \cong Rf$ and in fact this isomorphism is given by right multiplication by any nonzero element of eRf or fRe.*

PROOF. Let a, b be elements of R such that eaf and fbe are nonzero. Define $\alpha: Re \to Rf$ by $re \mapsto reaf$. Then $Re \oplus Reaf$ is a finitely-generated submodule of $R \oplus R$ and so is quasiprojective since R is a left semi-HQ-ring. Since α is an epimorphism of Re onto $Reaf$ and Re is projective, it follows from Lemma 1.1 of [5] that $Reaf$ is projective and that α is a retraction. Since Re is indecomposable, α is then a monomorphism.

Similarly the homomorphism $\beta: Rf \to Re$ given by $rf \mapsto rfbe$ is in fact a monomorphism. We are therefore done if we can show that $Re = \operatorname{im}(\alpha\beta) = Reafbe$ for then β would be an epimorphism and hence an isomorphism.

But we have a descending sequence $Re \supseteq Re\alpha\beta \supseteq Re(\alpha\beta)^2 \supseteq \cdots$ of principal left ideals of R which must terminate since R is right perfect and therefore satisfies the descending chain condition on principal left ideals. Therefore there exists an integer n such that $Re(\alpha\beta)^n = Re(\alpha\beta)^{n+1}$. In particular, $e(\alpha\beta)^n = re(\alpha\beta)^{n+1}$ for some $r \in R$ and so $e - re\alpha\beta \in \ker(\alpha\beta)^n$. But α and β are both monomorphisms so this kernel equals 0. Thus $e = re\alpha\beta$ and so $Re = Re\alpha\beta$. $\square$

(2.6) COROLLARY. *If R is a right perfect left semi-HQ-ring and e is an indecomposable idempotent of R then eRe is a division ring.*

PROOF OF THEOREM D. Let R be a right perfect left semi-HQ-ring. Then there exists a set $\{e_1, \cdots, e_n\}$ of indecomposable orthogonal idempotents of R such that $R = \bigoplus_{i=1}^n Re_i$ and each $Re_i/J(R)e_i$ is a simple left R-module. Furthermore, we have the Z-decomposition

$$R = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{n} e_i R e_j.$$

Fix some index k, $1 \leqq k \leqq n$, and let $S_k = \sum e_i R e_j$ where the sum ranges over all indices i, j such that $Re_i \cong Re_j \cong Re_k$. Then S_k is a subring of R which is isomorphic to the full matrix ring $(e_k R e_k)_t$, where t is the number of different indices i in the above sum. Since $e_k R e_k$ is a division ring by Corollary 2.6, S_k is a simple ring.

Now let $S = \sum_{k=1}^n S_k$. Then this is a subring of R and, since each S_k is simple, S is completely reducible and contains $1 = e_1 + \cdots + e_n$.

Let $T = \sum e_i R e_j$ where the sum ranges over all indices i, j such that

Re_i and Re_j are not isomorphic. We will be done if we can show that $T=J(R)$. Clearly $T\subseteq J(R)$ since we have the chain of implications $e_iae_j\notin J(R)\Rightarrow Re_iae_j\not\subset J(R)e_j\Rightarrow Re_iae_j=Re_j\Rightarrow$ the homomorphism $Re_i\to Re_j$ given by $re_i\mapsto re_iae_j$ is an isomorphism $\Rightarrow e_iae_j\notin T$. Since S is completely reducible and $R=S+T$, to show that $T=J(R)$ it therefore suffices to show that T is a two-sided ideal of R. T is clearly closed under addition. Let $e_iae_j\in T$ and consider $0\neq y=e_kbe_iae_j$. If Re_k is not isomorphic to Re_j then $y\in e_kRe_j\subseteq T$. Hence assume that $Re_k\cong Re_j$. If $\beta:Re_k\to Re_i$ and $\alpha:Re_i\to Re_j$ are the R-homomorphisms respectively given by $re_k\mapsto re_kbe_i$ and $re_i\mapsto re_iae_j$ then by the same reasoning as in Lemma 2.5 each of these maps is a monomorphism. But $\beta\alpha$ is just the map given by right multiplication by y and so, by Lemma 2.5, is an isomorphism between Re_k and Re_j. Therefore α must also be an epimorphism and so is an isomorphism between Re_i and Re_j, a contradiction. Hence $y=0$ for $Re_k\cong Re_j$ and so $RT\subseteq T$. A similar proof shows $TR\subseteq T$. $\square$

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Institute of Mathematics, The Hebrew University of Jerusalem, Jerusalem, Israel

Current address: Department of Mathematics, University of Florida, Gainesville, Florida 32601