CHARACTERIZATION OF RINGS USING QUASIPROJECTIVE MODULES. III

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Abstract. A ring \( R \) is regular [completely reducible] if and only if the character module of every left \( R \)-module is quasi-injective [quasiprojective]. Submodules of quasiprojective left \( R \)-modules over a left perfect ring \( R \) are quasiprojective if and only if singular left \( R \)-modules are injective. A splitting theorem for right perfect rings over which submodules of quasiprojective left \( R \)-modules are quasiprojective is also proven. These results continue the author’s previous work ([5] and [6]).

A left \( R \)-module \( M \) is quasiprojective if and only if, for every epimorphism \( \alpha : _RM \to _RN \), \( \text{Hom}_R (M, N) = \text{Hom}_R (M, M) \alpha \). Dually, \( M \) is quasi-injective if and only if, for every monomorphism \( \beta : _RN \to _RM \), \( \text{Hom}_R (N, M) = \beta \text{Hom}_R (M, M) \). In [5] and [6] we showed how quasiprojective modules can be used to characterize the rings over which they are defined. Here we will prove additional ring characterizations using quasiprojective and quasi-injective modules.

In what follows \( R \) will always denote an associative ring with identity and modules and morphisms will always be taken from the category \( R\text{-mod} \) [resp. \( \text{mod}-R \)] of left [resp. right] unitary \( R \)-modules. Morphisms will always be written acting on the side opposite ring multiplication.

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1. Character modules. Let \( G \) be an injective cogenerator over the ring \( \mathbb{Z} \) of integers and let \( \chi = \text{Hom}_\mathbb{Z} ( , G) \) be the \( G \)-character functor. Then \( \chi \) can be considered as a faithful exact contravariant functor \( R\text{-mod} \to \text{mod}-R \) [or \( \text{mod}-R \to R\text{-mod} \)] which commutes with finite direct sums. Furthermore, for any \( R \)-module \( M \) we have a canonical
embedding $M \to \chi^2(M)$. The module $\chi(M)$ is called the character module of $M$ (with respect to $G$). The major results of this section are:

**Theorem A.** A ring $R$ is regular (in the sense of von Neumann) if and only if the character module of every left $R$-module is quasi-injective.

**Theorem B.** The following are equivalent for a ring $R$:
1. $R$ is completely reducible.
2. $\chi(M)$ is quasiprojective for every right (left) $R$-module $M$.
3. $\chi^2(M)$ is quasiprojective for every left (right) $R$-module $M$.

We will adopt the following notation from [11]: If $M$ is a left $R$-module, then

- $\pi^{-1}(M)$ will denote the class of all epimorphisms $\alpha:RU\to RV$ such that $\text{Hom}(\text{id}_M, \alpha):\text{Hom}_R(M, U)\to \text{Hom}_R(M, V)$ is an epimorphism.
- $\iota^{-1}(M)$ will denote the class of all monomorphisms $\beta:RU\to RV$ such that $\text{Hom}(\beta, \text{id}_M):\text{Hom}_R(V, M)\to \text{Hom}_R(U, M)$ is an epimorphism.
- $\tau^{-1}(M)$ will denote the class of all monomorphisms $\gamma:UR\to VR$ such that $\gamma \otimes \text{id}_M: U \otimes R M \to V \otimes R M$ is a monomorphism.

Let $M$ be a left $R$-module and $U$ a right $R$-module. As a special case of the homological duality formulae [3, Propositions II,5.2 and II,5.2'] we have the canonical isomorphisms

$$\text{Hom}_R(U, \chi(M)) \cong \chi(U \otimes R M) \cong \text{Hom}_R(M, \chi(U))$$

from which we can immediately infer:

**Lemma.** If $M$ is a left $R$-module and $\alpha:UR\to VR$ is a homomorphism, then the following are equivalent:
1. $\alpha \in \pi^{-1}(M)$.
2. $\alpha \in \iota^{-1}(\chi(M))$.
3. $\chi(\alpha) \in \tau^{-1}(M)$.

(See [11, Proposition 3.2].)

If $M$ is a finitely-presented left $R$-module and $N$ is an arbitrary left $R$-module, then as a special case of the isomorphism given in [1, p. 63, Exercise 14] we have a canonical isomorphism

$$\chi(N) \otimes_R M \cong \chi(\text{Hom}_R(M, N)).$$

From this and Lemma 1.1 we then have:

**Lemma.** Let $M$ be a finitely-presented left $R$-module and let $\beta:RU\to RV$ be an epimorphism. Then the following are equivalent:
1. $\beta \in \tau^{-1}(M)$.
2. $\chi(\beta) \in \tau^{-1}(\chi(M))$.
3. $\chi^2(\beta) \in \tau^{-1}(M)$.
Corollary. If $M$ is a finitely-presented left $R$-module and $\chi(M)$ is quasi-injective then $M$ is quasiprojective.

Proof. If $\alpha : R\to R'$ is an epimorphism then $\chi(\alpha) : \chi(V)_{R'} \to \chi(M)_{R'}$ is a monomorphism and so, by the quasi-injectivity of $\chi(M)$, belongs to $\pi^{-1}(\chi(M))$. By Lemma 1.2, $\alpha$ then belongs to $\pi^{-1}(M)$, proving $M$ quasiprojective.

We are now in a position to prove our main theorems.

Proof of Theorem A. If $R$ is regular then every left $R$-module is flat. Lambek [9, p. 131] has shown that a left $R$-module $M$ is flat if and only if $\chi(M)$ is injective so, in particular, $\chi(M)$ is clearly quasi-injective.

Conversely assume that $R$ is not regular. Then there exists a left $R$-module $M$ which is not flat; that is to say, there exist right $R$-modules $U_R$ and $V_R$ together with a monomorphism $\lambda : U_R \to V_R$ such that $\lambda \otimes \text{id}_M : U_R \otimes_R M \to V_R \otimes_R M$ is not a monomorphism. Let $T = \chi(V) \otimes M$. Then $\chi(T) \cong \chi(V) \oplus \chi(M)$ and we have the canonical embeddings $\rho : V \to \chi(V) \to \chi(V) \oplus \chi(M) \cong \chi(T)$ and $\sigma : M \to T$.

If $0 \neq \sum_{i=1}^n (u_i \otimes m_i) \in \ker (\lambda \otimes \text{id}_M)$ then $0 \neq \sum_{i=1}^n (u_i \otimes m_i) \in U \otimes_R T$ and

$$\left[ \sum_{i=1}^n (u_i \otimes m_i) \right] (\lambda \rho \otimes \text{id}_T) = \left[ \sum_{i=1}^n (u_i \otimes m_i) \right] (\lambda \otimes \text{id}_M)(\rho \otimes \sigma) = 0,$$

proving that $\lambda \rho \otimes \text{id}_T : U \otimes_R T \to \chi(T) \otimes_R T$ is not a monomorphism, i.e., $\lambda \rho \notin \pi^{-1}(T)$. By Lemma 1.1, this means that $\lambda \rho \notin \pi^{-1}(\chi(T))$, proving that $\chi(T)$ is not quasi-injective.

Proof of Theorem B. The implications (1)$\Rightarrow$ (2)$\Rightarrow$ (3) are trivial. Therefore assume (3). If $N$ is an injective left $R$-module then $N$ can be canonically embedded in $\chi^2(N)$ and so is a direct summand of $\chi^2(N)$. Since $\chi^2(N)$ is quasiprojective, so is $N$ and so we have that every injective left $R$-module $M$ is quasiprojective, which proves that $R$ is a quasi-frobenius ring [2, Corollary 2.3]. In particular, $R$ is left noetherian.

Every finitely-generated left $R$-module over a left noetherian ring is finitely-presented [1, p. 36]. Since $R$ is completely reducible if every finitely-generated left $R$-module is quasiprojective (the proof is the same as that of Theorem 1.3 of [5]) it suffices by Corollary 1.3 to show that $\chi(M)$ is quasi-injective for every finitely-generated left $R$-module $M$. Let $\alpha : U_R \to \chi(M)_R$ be a monomorphism. Then $\chi(\alpha) : \chi^2(M) \to \chi^2(U)$ is an epimorphism. But $\chi^2(M) \oplus \chi^2(U) \cong \chi^2(\chi^2(M) \oplus \chi^2(U))$ which is quasiprojective. Since a sufficient condition for an epimorphism $R \to Y$ to be a retraction is that $X \oplus Y$ be quasiprojective [6, Lemma 2.1], it follows that $\chi^2(\alpha)$ is a retraction and hence surely belongs to $\pi^{-1}(M)$. By Lemma
1.2, $\chi(\alpha)$ therefore belongs to $\tau^{-1}(M)$ and so Lemma 1.1 implies that $\alpha$ belongs to $r^{-1}(\chi(M))$. This proves that $\chi(M)$ is quasi-injective. □

A submodule $N$ of a left $R$-module $M$ is pure if and only if, for every right $R$-module $T$, the canonical imbedding $N \to M$ belongs to $\tau^{-1}(T)$. An $R$-module $U$ is pure-injective if and only if, for every pure submodule $N$ of an $R$-module $M$, the embedding $N \to M$ belongs to $r^{-1}(U)$.

If as our injective cogenerator $G$ we take as a special case $R/\mathbb{Z}$, where $R$ is the field of reals, then $\chi^2(M)$ is a compact $R$-module and $M$ is a pure submodule of $\chi^2(M)$. Moreover, a module $U$ is pure-injective if and only if it is a summand of $\chi^2(U)$ [12, pp. 706–708]. We then get the following corollary to Theorem B, generalizing a result of Griffith [7]:

(1.4) Corollary. A ring $R$ is completely reducible if and only if every pure-injective left $R$-module is quasiprojective.

2. Heredity with respect to quasiprojectives. In [6] we looked briefly at rings $R$ satisfying the condition that every [finitely-generated] submodule of a quasiprojective left $R$-module be quasiprojective. Such rings we will call left [semi-] HQ-rings. This condition is stronger than left hereditary and in fact we have shown [6, Theorem 5.1] that every factor ring of a left [semi-] HQ-ring is left [semi-] hereditary. If $R$ is left perfect then the converse also holds. We will now characterize these rings in more detail; our main results are:

Theorem C. A left perfect ring $R$ is a left HQ-ring if and only if every singular left $R$-module is injective.

Theorem D. If $R$ is a right perfect left semi-HQ ring then $R$ admits a splitting $R=S\oplus J(R)$ over $\mathbb{Z}$, where $S$ is a completely reducible subring of $R$ containing 1.

A two-sided ideal $I$ of a ring $R$ is left $T$-nilpotent if and only if, for every sequence $(a_i)$ of elements of $I$, there exists an integer $n$ such that $a_1 \cdot \ldots \cdot a_n=0$. Ideals satisfying this condition were first studied by Levitzki [10]. The following lemma is known:

(2.1) Lemma. Let $I$ be a left $T$-nilpotent two-sided ideal of a ring $R$. Then for every left $R$-module $M$, $IM$ is small in $M$.

Proof. $RU$ is small in $RV$ if and only if $U+RU=V$ implies $W=V$. Assume $IM$ is not small in $M$. We will arrive at a contradiction by constructing a sequence $(a_i)$ of elements of $I$ such that $a_1 \cdot \ldots \cdot a_n \neq 0$ for all $n$.

Since $IM$ is not small in $M$ there exists a proper submodule $N$ of $M$ with $IM+N=M$. In particular there exists $a_i \in I$ and $m_i \in M$ such...
that \(a_1m_1 \notin N\) (and hence surely \(a_1m_1 \neq 0\)). Now assume inductively that there exists \(a_1, \ldots, a_n \in I\) and \(m_n \in M\) such that \(a_1 \cdot \ldots \cdot a_n m_n \notin N\). Then \(m_n = y + \sum_{j=1}^{k} b_j x_j\) for \(y \in N\), \(b_j \in I\), and \(x_j \in M\). Since \(a_1 \cdot \ldots \cdot a_n m_n \notin N\) there exists some \(j\) such that \(a_1 \cdot \ldots \cdot a_n b_j x_j \notin N\). Take \(a_{n+1} = b_j\) and \(m_{n+1} = x_j\). □

(2.2) Proposition. Let \(R\) be a left HQ-ring and let \(H\) be a left T-nilpotent two-sided ideal of \(R\). Then \(H^2 = 0\). In particular, if \(R\) is left perfect then \(J(R)^2 = 0\).

Proof. Since \(H\) is left T-nilpotent, \(H = HR\) is small in \(R\) and so \(H\) is contained in the Jacobson radical \(J(R)\) of \(R\) (which equals the union of all small left ideals of \(R\)). Let \(S = R/H^2\). Since \(R\) is a left HQ-ring, \(R/H^2\) is left hereditary and so, in particular, \(J(R)/H^2\) is a projective left \(S\)-module since it is a left ideal of \(S\). Let \(sU = J(R)/H^3\) and \(sV = H^2/H^3\). Then \(U/V\) is \(S\)-isomorphic to \(J(R)/H^2\) and so is projective over \(S\). But \(V \subseteq HU\) and so is small in \(U\). Thus by the splitting of the exact sequence \(0 \rightarrow V \rightarrow U \rightarrow U/V \rightarrow 0\) over \(S\) we see that \(V\) is a small direct summand of \(U\) and so \(V = 0\). This proves that \(H^2 = H^3\). But by Lemma 2.1, \(H^3\) is small in \(H^2\) and so \(H^2 = 0\).

The second part of the theorem comes from the well-known fact that the Jacobson radical of a left perfect ring is left T-nilpotent. □

(2.3) Corollary. Let \(R\) be a left hereditary left perfect ring. Then the following are equivalent:

1. \(R\) is a left HQ-ring.
2. \(J(R)^2 = 0\).

Proof. (1)⇒(2) follows directly from Proposition 2.2. (2)⇒(1): Harada has shown that if \(R\) is left hereditary and semi-primary and if \(k\) is the index of nilpotency of \(J(R)\) then for any two-sided ideal \(I\) of \(R\), \(\text{l.gl. dim} (R/I) \leq k - 1\) [8, Theorem 6]. In particular, (2) implies that \(R/I\) is left hereditary for every two-sided ideal \(I\) of \(R\) and so \(R\) is a left HQ-ring since it is left perfect. □

Recall that a submodule \(N\) of a module \(M\) is large if and only if it has a nonzero intersection with any nonzero submodule of \(M\) and that the singular submodule \(Z(M)\) of a left \(R\)-module \(M\) is given by \(Z(M) = \{m \in M \mid (0:m)\text{ is a large left ideal of } R\}\). An \(R\)-module \(M\) is singular if and only if \(Z(M) = M\) and nonsingular if and only if \(Z(M) = 0\). Both the class of singular \(R\)-modules and the class of nonsingular \(R\)-modules are closed under taking submodules.

Proof of Theorem C. Let \(R\) be a left perfect ring and assume that \(R\) is a left HQ-ring as well. Let \(M\) be a singular left \(R\)-module. To show that...
$M$ is injective it suffices to show that every diagram of the form

\[
\begin{array}{c}
0 \rightarrow RH \rightarrow R \\
\downarrow \alpha \quad \beta \\
M
\end{array}
\]

can be completed commutatively, where $H$ is a large left ideal of $R$, $\lambda$ is the canonical embedding, and $\alpha$ is an arbitrary nonzero homomorphism [9, p. 93].

Let $K = \ker(\alpha)$ and suppose that $K$ is not large in $R$. Then there exists a nonzero left ideal $I$ of $R$ such that $K \cap I = 0$. Since $H$ is large in $R$, $I' = H \cap I$ is nonzero and $I' \cap K = 0$. Thus the restriction of $\alpha$ to $I'$ is a monomorphism and so $I' \cong I' \alpha$ which is a submodule of a singular module and so is singular. On the other hand, $R$ is in particular left hereditary and so is nonsingular. Therefore $I'$, a submodule of $R$, is also nonsingular, yielding a contradiction which proves that $K$ is large in $R$.

By Proposition 2.2, $J(R)^2 = 0$ and so $J(R)$ is a left $R/J(R)$-module and so is completely reducible for, by the left perfectness of $R$, $R/J(R)$ is a completely reducible ring. Therefore $J(R) \subseteq \text{soc}(R)$. But the socle of a left $R$-module is precisely the intersection of all of its large submodules and so, in particular, $J(R) \subseteq K$. Therefore $R/K$ is a left $R/J(R)$-module and thus is completely reducible.

The map $\alpha$ induces a monomorphism $\tilde{\alpha} : H/K \rightarrow M$. Since $R/K$ is a completely reducible $R$-module, $H/K$ is a direct summand of $R/K$ and so we have a canonical projection $\pi : R/K \rightarrow H/K$. Finally, let $\nu : R \rightarrow R/K$ be the canonical epimorphism. Then $\nu \pi \tilde{\alpha}$ is an $R$-homomorphism $R \rightarrow M$ and for every $h \in H$, $h \nu \pi \tilde{\alpha} = (h+K) \pi \tilde{\alpha} = (h+K) \tilde{\alpha} = h \alpha$ so $\beta = \nu \pi \tilde{\alpha}$ is the homomorphism we seek.

Conversely, assume that $R$ is a left perfect ring over which every singular module is injective. Let $I$ be a two-sided ideal of $R$ and let $S = R/I$. For $M$ a left $S$-module let $\bar{M}$ be the injective hull of $M$ over $S$. Then $\bar{M}/M$ is large in $\bar{M}$ and so, as is easily seen, $\bar{M}/M$ is a singular left $R$-module. Therefore $\bar{M}/M$ is injective over $R$ and hence injective over $S$. Since the sequence $0 \rightarrow M \rightarrow \bar{M} \rightarrow \bar{M}/M \rightarrow 0$ is exact over $S$, $\text{inj dim } (\bar{M}/M) \leq 1$. Since this is true for any left $S$-module $M$, $S$ is left hereditary [3, p. 112] and so $R$ is a left HQ-ring by Theorem 5.1 of [6]. □

Note that the argument in the above proof is based on the techniques of [4].

(2.4) Corollary. If $R$ is a left perfect left HQ-ring then $Z(M)$ is a direct summand of $M$ for every left $R$-module $M$.
We now turn to look at right perfect left semi-$H\!Q$-rings, and use techniques based on those of Zaks [13] to obtain a classical splitting theorem (Theorem D).

(2.5) LEMMA. Let $R$ be a right perfect left semi-$H\!Q$-ring. If $e$ and $f$ are indecomposable idempotents of $R$ with $eRe$ and $fRe$ nonzero, then $Re \cong Rf$ and in fact this isomorphism is given by right multiplication by any nonzero element of $eRe$ or $fRe$.

PROOF. Let $a$, $b$ be elements of $R$ such that $eaf$ and $fbe$ are nonzero. Define $\alpha: Re \rightarrow Rf$ by $re \mapsto eaf$. Then $Re \oplus Reaf$ is a finitely-generated submodule of $R \oplus R$ and so is quasiprojective since $R$ is a left semi-$H\!Q$-ring. Since $\alpha$ is an epimorphism of $Re$ onto $Reaf$ and $Re$ is projective, it follows from Lemma 1.1 of [5] that $Reaf$ is projective and that $\alpha$ is a retraction. Since $Re$ is indecomposable, $\alpha$ is then a monomorphism.

Similarly the homomorphism $\beta: Rf \rightarrow Re$ given by $rf \mapsto rbe$ is in fact a monomorphism. We are therefore done if we can show that $Re = \text{im } (\alpha \beta) = Reafbe$ for then $\beta$ would be an epimorphism and hence an isomorphism.

But we have a descending sequence $Re \supseteq Re\alpha \beta \supseteq Re(\alpha \beta)^2 \supseteq \cdots$ of principal left ideals of $R$ which must terminate since $R$ is right perfect and therefore satisfies the descending chain condition on principal left ideals. Therefore there exists an integer $n$ such that $Re(\alpha \beta)^n = Re(\alpha \beta)^{n+1}$. In particular, $e(\alpha \beta)^n = re(\alpha \beta)^{n+1}$ for some $r \in R$ and so $e - rex \beta \in \ker (\alpha \beta)^n$. But $\alpha$ and $\beta$ are both monomorphisms so this kernel equals 0. Thus $e = rex \beta$ and so $Re = Re \alpha \beta$. □

(2.6) COROLLARY. If $R$ is a right perfect left semi-$H\!Q$-ring and $e$ is an indecomposable idempotent of $R$ then $eRe$ is a division ring.

PROOF OF THEOREM D. Let $R$ be a right perfect left semi-$H\!Q$-ring. Then there exists a set $\{e_1, \cdots , e_n\}$ of indecomposable orthogonal idempotents of $R$ such that $R = \bigoplus_{i=1}^n Re_i$ and each $Re_i/J(R)e_i$ is a simple left $R$-module. Furthermore, we have the $Z$-decomposition

$$R = \bigoplus_{i=1}^n \bigoplus_{j=1}^n e_i Re_j.$$ 

Fix some index $k$, $1 \leq k \leq n$, and let $S_k = \sum e_i Re_i$ where the sum ranges over all indices $i$, $j$ such that $Re_j \subseteq Re_i \subseteq Re_k$. Then $S_k$ is a subring of $R$ which is isomorphic to the full matrix ring $(e_k Re_k)_t$, where $t$ is the number of different indices $i$ in the above sum. Since $e_k Re_k$ is a division ring by Corollary 2.6, $S_k$ is a simple ring.

Now let $S = \sum_{k=1}^n S_k$. Then this is a subring of $R$ and, since each $S_k$ is simple, $S$ is completely reducible and contains $1 = e_1 + \cdots + e_n$.

Let $T = \sum e_i Re_j$ where the sum ranges over all indices $i$, $j$ such that
$Re_i$ and $Re_j$ are not isomorphic. We will be done if we can show that $T=J(R)$. Clearly $T \subseteq J(R)$ since we have the chain of implications $e_i ae_j \in J(R) \Rightarrow Re_i ae_j \subseteq J(R) e_i \Rightarrow Re_i ae_j = Re_j \Rightarrow$ the homomorphism $Re_i \rightarrow Re_j$ given by $re_i \mapsto re_i ae_j$ is an isomorphism $\Rightarrow e_i ae_j \notin T$. Since $S$ is completely reducible and $R = S + T$, to show that $T = J(R)$ it therefore suffices to show that $T$ is a two-sided ideal of $R$. $T$ is therefore closed under addition. Let $e_i ae_j \in T$ and consider $0 \neq y = e_i be_i ae_j$. If $Re_k$ is not isomorphic to $Re_j$ then $y \in e_j Re_k \subseteq T$. Hence assume that $Re_k \cong Re_j$. If $\beta: Re_k \rightarrow Re_i$ and $\alpha: Re_i \rightarrow Re_j$ are the $R$-homomorphisms respectively given by $re_k \mapsto re_k be_i$ and $re_i \mapsto re_i ae_j$ then by the same reasoning as in Lemma 2.5 each of these maps is a monomorphism. But $\beta \alpha$ is just the map given by right multiplication by $y$ and so, by Lemma 2.5, is an isomorphism between $Re_k$ and $Re_j$. Therefore $\alpha$ must also be an epimorphism and so is an isomorphism between $Re_i$ and $Re_j$, a contradiction. Hence $y = 0$ for $Re_k \cong Re_j$ and so $RT \subseteq T$. A similar proof shows $TR \subseteq T$. □

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