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EQUIVALENT TYPES OF INVARIANT MEANS ON LOCALLY COMPACT GROUPS

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ABSTRACT. For G a locally compact amenable group, we establish the equivalence of left invariant means and topologically left invariant means on $L^\infty(G)$.

1. **Introduction and notation.** Let G be a locally compact group with left Haar measure μ. Let $L^1(G)$ and $L^\infty(G)$ denote the usual Banach function spaces on G. $L^1(G)$ is a Banach *-algebra under the convolution operation

$$x * y(g) = \int x(h)y(h^{-1}g)\, d\mu(h)$$

and the adjoint map

$$x^*(g) = \Delta(g^{-1})\overline{x(g^{-1})}$$

where Δ is the modular function on G. A *weight* on G is a nonnegative function $x \in L^1(G)$ such that $\int x(g)\, d\mu(g) = 1$. Denote by P the set of all weights on G and observe that P is a semigroup under convolution. For f a complex-valued function on G, define ${}_gf$ $(g \in G)$ by ${}_gf(h) = f(g^{-1}h)$.

A linear functional m on $L^\infty(G)$ is called a *mean* if

(i) $m(f) \geqq 0$ for all $f \in L^\infty(G)$, $f \geqq 0$ and

(ii) $m(1) = 1$ where 1 denotes the identity function.

Clearly every weight x in P defines a mean on $L^\infty(G)$ via $\langle x, f\rangle = \int x(g)f(g)\, d\mu(g)$.

Let m be a mean on $L^\infty(G)$. m is called a *left invariant mean* (LIM) if

$$m({}_gf) = m(f) \quad \text{for all } f \in L^\infty(G), g \in G.$$

m is called a *topologically left invariant mean* (TLIM) if

$$m(x * f) = m(f) \quad \text{for all } f \in L^\infty(G),\ x \in P,$$

or equivalently

$$m(x * f) = m(f) \int x(g)\, d\mu(g) \quad \text{for all } f \in L^\infty(G),\ x \in L^1(G).$$

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G is called *amenable* if there exists a LIM on $L^\infty(G)$. TLIM's were introduced by Hulanicki [4] as a natural extension of LIM's to non-discrete groups. Among other results he proved that every TLIM is also a LIM. Subsequently, Namioka [5] showed that the existence of a LIM implies the existence of a TLIM. The purpose of this note is to show that in fact every LIM is also a TLIM. This answers a question raised by Greenleaf (see [3, Lemma 2.2.2 and remarks]).

2. **Equivalence of LIM's and TLIM's.** We shall now prove the following

THEOREM. *Let G be an amenable group, m a LIM on $L^\infty(G)$. Then m is a TLIM.*

For $f \in L^\infty(G)$ and $x \in L^1(G)$, left invariance of m gives

$$m({}_g x) * f) = m({}_g(x * f)) = m(x * f) \quad \text{for all } g \in G.$$

Hence $x \to m(x * f)$ is a left invariant bounded linear functional on $L^1(G)$ so that there exists a constant $k(f)$ such that

$$m(x * f) = k(f) \int x(g)\, d\mu(g) \quad \text{for all } x \in L^1(G).$$

It is immediate that k is a mean on $L^\infty(G)$. Further if $x \in P$ then $x * x \in P$ so that

$$k(x * f) = m(x * (x * f)) = m((x * x) * f) = k(f)$$

and k is a TLIM. The theorem will be proved if we show that $m = k$.

Fix $f \in L^\infty(G)$. Choose a net $\{x_\gamma\}_{\gamma \in \Omega} \subset P$ such that $w^* - \lim_\gamma x_\gamma = m$ and define F_γ on G by $F_\gamma(g) = \langle {}_g x_\gamma, f \rangle$. Left invariance of m means that $F_\gamma \to m(f)$ pointwise on G. The theorem will follow from the following

LEMMA. *$F_\gamma \to m(f)$ almost uniformly on every compact subset of G.*

PROOF. If we were dealing with sequences rather than nets, then the lemma would be a trivial application of Egoroff's theorem. With nets, however, a little delicacy is required.

Let K be a compact set with $\mu(K) > 0$. For k a positive integer, $\gamma \in \Omega$, define

$$E_{k,\gamma} = \bigcap_{\gamma' \geqq \gamma} \{g \in K : |F_{\gamma'}(g) - m(f)| \leqq 1/k\}.$$

Since F_γ is continuous, $E_{k,\gamma}$ is a compact subset of K. Note that for fixed k, $\{E_{k,\gamma}\}$ is an increasing net (in the sense that $\gamma \geqq \gamma' \Rightarrow E_{k,\gamma} \supseteq E_{k,\gamma'}$) with $\bigcup_\gamma E_{k,\gamma} = K$. Let χ_K, $\chi_{E_{k,\gamma}}$ be the characteristic functions of K and $E_{k,\gamma}$ respectively. We then have that $\{\chi_{E_{k,\gamma}}\}$ is a bounded monotone increasing net in $L^\infty(G)$ for each k and $\chi_K = \sup_\gamma \chi_{E_{k,\gamma}}$. Now $L^\infty(K)$ may be regarded

as a W^*-algebra on the Hilbert space $L^2(K)$. The predual of $L^\infty(K)$ is $L^1(K)$ so that every nonnegative element in $L^1(K)$ is a *normal* positive linear functional on $L^\infty(K)$ [**1**, Chapitre 1, §§3 and 4]. Hence $\langle \chi_K, \chi_K\rangle = \sup_\gamma \langle \chi_K, \chi_{E_{k,\gamma}}\rangle$ or,

$$\lim_\gamma \mu(E_{k,\gamma}) = \mu(K) \quad \text{for each } k.$$

Fix $\varepsilon>0$. For each k, choose γ_k such that $\mu(K\backslash E_{k,\gamma_k})<\varepsilon/2^k$ and let $E_0=\bigcap_k E_{k,\gamma_k}$. E_0 is a compact set and

$$\mu(K\backslash E_0) = \mu\left(\bigcup_k K\backslash E_{k,\gamma_k}\right) \leqq \sum_k \mu(K\backslash E_{k,\gamma_k}) < \varepsilon.$$

Finally it is clear that $F_\gamma \to m(f)$ uniformly on E_0.

It should be noted that the above theorem resembles somewhat the condition (FC*) of [**2**]. Using a technique similar to the one employed in the proof of Lemma 1.4.3 of [**2**], we could show directly that the above lemma implies that $F_\gamma \to m(f)$ *uniformly* on all compact sets.

PROOF OF THEOREM. By the above lemma we can find a compact set E with $\mu(E)>0$ such that $F_\gamma \to m(f)$ uniformly on E. Therefore

$$\lim_\gamma \int_E F_\gamma(g)\, d\mu(g) = m(f)\mu(E).$$

But

$$\begin{aligned}\int_E F_\gamma(g)\, d\mu(g) &= \int_G \chi_E(g)\left[\int_G x_\gamma(g^{-1}h)f(h)\, d\mu(h)\right] d\mu(g)\\ &= \int_G (\chi_E * x_\gamma)(h)f(h)\, d\mu(h)\\ &= \langle \chi_E * x_\gamma, f\rangle = \langle x_\gamma, \chi_E^* * f\rangle\end{aligned}$$

so that $\lim_\gamma \int_E F_\gamma(g)\, d\mu(g) = m(\chi_E^* * f)$. Hence

$$\begin{aligned}m(f)\mu(E) &= m(\chi_E^* * f)\\ &= k(f)\int \chi_E^*(g)\, d\mu(g) = k(f)\mu(E).\end{aligned}$$

Therefore $m(f)=k(f)$ and f being arbitrary, $m=k$. Hence m is a TLIM.

The notion of LIM and TLIM may be applied to $CB(G)$—the space of bounded continuous functions on G. The method above may be applied here to show that on $CB(G)$ every LIM is again a TLIM.

REFERENCES

1. J. Diximier, *Les algèbres d'opérateurs dans l'espace Hilbertien*, Gauthier-Villars, Paris, 1969.

2. W. R. Emerson and F. P. Greenleaf, *Covering properties and Følner conditions for locally compact groups*, Math. Z. **102** (1967), 370–384. MR **36** #3912.

3. F. P. Greenleaf, *Invariant means on topological groups and their applications*, Van Nostrand Math. Studies, no. 16, Van Nostrand, New York, 1969. MR **40** #4776.

4. A. Hulanicki, *Means and Følner condition on locally compact groups*, Studia Math. **27** (1966), 87–104. MR **33** #4178.

5. I. Namioka, *On a recent theorem by H. Reiter*, Proc. Amer. Math. Soc. **17** (1966), 1101–1102. MR **34** #4406.

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