

MULTIPLIER OPERATORS ON B^* -ALGEBRAS¹

B. D. MALVIYA AND B. J. TOMIUK

ABSTRACT. The purpose of this paper is to give a characterization of the dual B^* -algebra and the algebra of bounded linear operators on Hilbert space in terms of their multipliers.

1. All algebras and vector spaces under consideration are over the complex field C . If A is a Banach algebra, A^* will denote the first conjugate space and A^{**} the second conjugate space of A . For any Hilbert space H , $L(H)$ will denote the algebra of continuous linear operators on H .

Let A be a B^* -algebra. Following Máté [5], we call a bounded linear operator T mapping A into itself a multiplier if $T(xy) = x(Ty)$ for all $x, y \in A$. The set $M(A)$ of all multipliers on A is a Banach algebra. For every $a \in A$, the right multiplication operator T_a is a multiplier on A , $I_A = \{T_a : a \in A\}$ is a closed left ideal of $M(A)$ and the mapping $a \rightarrow T_a$ is an isometric anti-isomorphism of A onto I_A .

From [3, p. 869, Theorem 7.1] it follows that if A is a B^* -algebra then the two Arens products defined on A^{**} coincide. For later use we sketch one of the Arens products. We do this in stages as follows [1], [3]: Let $x, y \in A, f \in A^*, F, G \in A^{**}$.

- (i) Define $f \circ x$ by $(f \circ x)y = f(xy)$; $f \circ x \in A^*$.
- (ii) Define $G \circ f$ by $(G \circ f)x = G(f \circ x)$; $G \circ f \in A^*$.
- (iii) Define $F \circ G$ by $(F \circ G)f = F(G \circ f)$; $F \circ G \in A^{**}$.

For any Hilbert space H , $LC(H)$ will denote the subalgebra of $L(H)$ consisting of compact operators and $\tau c(H)$ the subalgebra of trace class operators on H . We shall denote the trace function of $\tau c(H)$ by $\text{tr}(\cdot)$ and the trace norm by $\tau(\cdot)$; $\tau(T) = \text{tr}((T^*T)^{1/2})$ for all $T \in \tau c(H)$. As a Banach space $\tau c(H)$ can be identified with the conjugate space of $LC(H)$ in the following way: For each continuous linear functional f on $LC(H)$ there exists a unique T in $\tau c(H)$ such that $f(S) = \text{tr}(ST)$ for $S \in LC(H)$ and $\|f\| = \tau(T)$ [7, p. 46, Theorem 1]. Similarly, the conjugate space of $\tau c(H)$ can be identified (isometrically) with $L(H)$ [7, p. 47, Theorem 2].

Received by the editors January 22, 1971.

AMS 1969 subject classifications. Primary 4650; Secondary 4655.

Key words and phrases. B^* -algebra, dual B^* -algebra, multiplier algebra, Arens product, compact operators, bounded linear operators, Hilbert space.

¹ This research was supported by the National Research Council of Canada.

Thus the second conjugate space of $LC(H)$ is isometrically isomorphic to $L(H)$. In fact, it can be shown that this isomorphism is actually a *-isomorphism when the second conjugate space of $LC(H)$ is given the Arens product.

Let $\{A_\lambda: \lambda \in \Lambda\}$ be a family of Banach algebras. Let $\sum A_\lambda$ be the set of all functions on Λ with $f(x) \in A_\lambda$, for each λ , and such that $\|f\| = \sup_\lambda \|f(\lambda)\| < \infty$. Then under the usual operations for functions and the norm $\|f\|$, $\sum A_\lambda$ is a Banach algebra. It is called the normed full direct sum of the algebras A_λ [6, p. 77]. Let $(\sum A_\lambda)_0$ be the subset of $\sum A_\lambda$ consisting of all f such that, for every $\varepsilon > 0$, the set $\{\lambda: \|f(\lambda)\| \geq \varepsilon\}$ is finite. Then $(\sum A_\lambda)_0$ is a closed subalgebra of $\sum A_\lambda$ [6, p. 107].

For any set S in a Banach algebra A , let $l(S)$ and $r(S)$ be the left and right annihilators of S , respectively. A is called dual if $l(r(J)) = J$ and $r(l(R)) = R$ for every closed left ideal J and every closed right ideal R of A . As usual $\text{cl}(S)$ will denote the closure of S in A .

2. We devote this section to several lemmas which will be useful to us in §3.

LEMMA 2.1. *To each multiplier T on the algebra $LC(H)$ there corresponds a unique element a_T in $L(H)$ such that $T(s) = sa_T$ for all $s \in LC(H)$; $\|T\| = \|a_T\|$. Thus the mapping $T \rightarrow a_T$ is an isometric anti-isomorphism of $M(LC(H))$ onto $L(H)$.*

PROOF. Let $A = LC(H)$ and let $T \in M(A)$. Since A has an approximate identity, by [5, p. 810, Theorem 1], there exists a unique element F in A^{**} such that

$$(1) \quad (F \circ f)s = f(T(s)) \quad (s \in A, f \in A^*).$$

For $s \in A$ and $f \in A^*$, let t_f and $t_{f \circ s}$ be the elements of $\tau c(H)$ such that $f(a) = \text{tr}(at_f)$ and $(f \circ s)(a) = \text{tr}(at_{f \circ s})$ for all $a \in A$. (See [7, p. 46, Theorem 1].) Since

$$\text{tr}(t_f sa) = f(sa) = (f \circ s)a = \text{tr}(t_{f \circ s} a) \quad (a \in A),$$

[7, p. 45, Lemma 1] shows that

$$(2) \quad t_{f \circ s} = t_f s \quad (s \in A, f \in A^*).$$

Thus

$$(3) \quad (F \circ f)s = F(f \circ s) = \text{tr}(t_{f \circ s} t_F) = \text{tr}(t_f s t_F) \quad (s \in A, f \in A^*),$$

where t_F is the unique element in $L(H)$ such that $F(f) = \text{tr}(t_f t_F)$ for all $f \in A^*$. (See [7, p. 47, Theorem 2].) But $f(T(s)) = \text{tr}(t_f T(s))$. Hence from (1) and (3) it follows that

$$f(T(s)) = \text{tr}(t_f s t_F) \quad (f \in A^*).$$

Recalling [7, p. 45, Lemma 1], we see that $T(s) = st_F$ for all $s \in A$. Taking $a_T = t_F$ completes the proof.

COROLLARY 2.2. *Let $A = LC(H)$. Then there exists an isometric anti-isomorphism ϕ of $M(A)$ onto A^{**} such that $\phi(I_A) = \pi(A)$, where $\pi(A)$ is the canonical image of A in A^{**} .*

PROOF. For each $a \in A$, the right multiplication operator T_a is a multiplier on A . Hence by Lemma 2.1, there exists $b \in L(H)$ such that $T_a x = xb$ for all $x \in A$. But this means that $xa = xb$, for all $x \in A$, which clearly implies that $a = b$. Let ϕ_1 be the mapping $T \rightarrow a_T$ of $M(A)$ onto $L(H)$ given in Lemma 2.1, and let ϕ_2 be the mapping $a \rightarrow F_a$ which identifies $L(H)$ with A^{**} ; ϕ_2 is an isometric $*$ -isomorphism of $L(H)$ onto A^{**} . Let ϕ be the composite map $\phi = \phi_2 \circ \phi_1$. Then ϕ is an isometric anti-isomorphism of $M(A)$ onto A^{**} such that $\phi(I_A) = \pi(A)$.

The following lemma is easy to prove and we state it mainly for convenience.

LEMMA 2.3. *Let $\{A_\lambda : \lambda \in \Lambda\}$ be a family of semisimple Banach algebras and let $A = (\sum A_\lambda)_0$. For each $\lambda \in \Lambda$, let $I_\lambda = \{f \in A : f(\mu) = 0 \text{ if } \mu \neq \lambda\}$ and $B_\lambda = \{f \in A : f(\lambda) = 0\}$. Then*

- (i) $I_\lambda \cap B_\lambda = (0)$ and $I_\lambda + B_\lambda = A$.
- (ii) $l(I_\lambda) = r(I_\lambda) = B_\lambda$ and $l(B_\lambda) = r(B_\lambda) = I_\lambda$.

LEMMA 2.4. *Let A, I_λ , and B_λ be as in Lemma 2.3. Let $T \in M(A)$. Then*

- (i) T leaves each I_λ invariant, i.e., $T(I_\lambda) \subset I_\lambda$.
- (ii) If T_λ denotes the restriction of T to I_λ , then

$$\|T\| = \sup_\lambda \|T_\lambda\|.$$

PROOF. (i) Let $x \in B_\lambda$ and $y \in I_\lambda$. Then $0 = T(xy) = xTy$ which shows that $Ty \in r(B_\lambda) = I_\lambda$ by Lemma 2.3. Hence $T(I_\lambda) \subset I_\lambda$.

(ii) Clearly $\|T_\lambda\| \leq \|T\|$ for all λ . Let $\varepsilon > 0$ be given. Then there exists $f \in A$, $\|f\| = 1$, such that $\|T\| - \varepsilon \leq \|Tf\|$. Since $A = (\sum A_\lambda)_0$, there exists $\lambda_1, \lambda_2, \dots, \lambda_n$ such that $\|f(\lambda_i)\| \geq \varepsilon/\|T\|$ and $\|f(\lambda)\| < \varepsilon/\|T\|$ for $\lambda \neq \lambda_i$ ($i = 1, \dots, n$). Let $g \in A$ be such that $g(\lambda_i) = f(\lambda_i)$ and $g(\lambda) = 0$ for $\lambda \neq \lambda_i$ ($i = 1, 2, \dots, n$). Then $\|f\| = \|g\|$ and

$$\|Tf\| = \|Tg\| = \sup_{1 \leq i \leq n} \|T_{\lambda_i}(g(\lambda_i))\|,$$

so that $\|Tf\| = \|T_{\lambda_{i_0}}(g(\lambda_{i_0}))\|$ for some $i_0, 1 \leq i_0 \leq n$. Since $\|T_{\lambda_{i_0}}(g(\lambda_{i_0}))\| \leq \|T_{\lambda_{i_0}}\|$, we have $\|T\| - \varepsilon \leq \|T_{\lambda_{i_0}}\|$. Hence $\|T\| = \sup_\lambda \|T_\lambda\|$.

LEMMA 2.5. *Let $\{A_\lambda : \lambda \in \Lambda\}$ be a family of semisimple Banach algebras and let $A = (\sum A_\lambda)_0$. Then $M(A)$ is isometrically isomorphic to the normed full direct sum of the algebras $M(A_\lambda)$.*

PROOF. For each $\lambda \in \Lambda$, let $I_\lambda = \{f \in A : f(\mu) = 0 \text{ if } \mu \neq \lambda\}$ and, for each $T \in M(A)$, let T_λ be the restriction of T to I_λ ; T_λ is a multiplier on I_λ . Since A_λ is isometrically isomorphic to I_λ , each T_λ may be identified as an element of $M(A_\lambda)$ with the same norm. For $T \in M(A)$, let \mathcal{F}_T be the function on Λ such that $\mathcal{F}_T(\lambda) = T_\lambda$. By Lemma 2.4, \mathcal{F}_T is an element of the normed full direct sum $\sum M(A_\lambda)$ with $\|\mathcal{F}_T\| = \|T\|$. Hence $T \rightarrow \mathcal{F}_T$ is an isometric isomorphism of $M(A)$ into $\sum M(A_\lambda)$. To show that this mapping is onto, let $\mathcal{F} \in \sum M(A_\lambda)$ and let T be the mapping on A such that $(Tf)(\lambda) = \mathcal{F}(\lambda)f(\lambda)$. It is easy to see that T is a multiplier on A with $\|T\| = \|\mathcal{F}\|$. Thus $T \rightarrow \mathcal{F}_T$ is onto and this completes the proof.

COROLLARY 2.6. *Let A be a dual B^* -algebra and let $\{I_\lambda : \lambda \in \Lambda\}$ be the family of all minimal closed two-sided ideals of A . For each $T \in M(A)$ and $\lambda \in \Lambda$, let T_λ be the restriction of T to I_λ . Let $M_\lambda = \{T_\lambda : T \in M(A)\}$. Then $M(A)$ is isometrically isomorphic to the normed full direct sum of the algebras M_λ .*

PROOF. By [6, p. 267, Theorem (4.10.14)], $A = (\sum I_\lambda)_0$ and so, by Lemma 2.5, $M(A)$ is isometrically isomorphic to the normed full direct sum $\sum M(I_\lambda)$. Now, since $I_\lambda \cap r(I_\lambda) = (0)$ and $I_\lambda + r(I_\lambda) = A$, it is easy to show that $M_\lambda = M(I_\lambda)$. Hence $M(A)$ is isometrically isomorphic to $\sum M_\lambda$.

3. We are now ready to prove the characterizations mentioned in the abstract.

THEOREM 3.1. *Let A be a B^* -algebra, A^{**} its second conjugate space and $\pi(A)$ the canonical image of A in A^{**} . Give A^{**} the Arens product. Then A is a dual algebra if and only if there exists an isometric anti-isomorphism ϕ of $M(A)$ onto A^{**} such that $\phi(I_A) = \pi(A)$.*

PROOF. Suppose that A is dual. Then there exists a family of Hilbert spaces $\{H_\lambda : \lambda \in \Lambda\}$ such that A is $*$ -isomorphic to $(\sum LC(H_\lambda))_0$ [4, p. 221, Lemma 2.3]. It now follows that A^* is isometrically isomorphic to $(\sum \tau c(H_\lambda))_1$, the L_1 -direct sum of the algebras $\tau c(H_\lambda)_1$, and that in turn A^{**} is isometrically isomorphic to the normed full direct sum $\sum L(H_\lambda)$ of the algebras $L(H_\lambda)$ [8, p. 532]. Letting $LC(H_\lambda) = A_\lambda$ and identifying A with $(\sum A_\lambda)_0$, Lemma 2.5 shows that $M(A)$ is isometrically isomorphic to the normed full direct sum of the algebras $M(A_\lambda)$. But, by Corollary 2.2, $M(A_\lambda)$ is isometrically anti-isomorphic to $L(H_\lambda)$, for each $\lambda \in \Lambda$. Hence $M(A)$ is isometrically anti-isomorphic to $\sum L(H_\lambda)$. Since $\sum L(H_\lambda)$ is $*$ -isomorphic to A^{**} , it follows that $M(A)$ is isometrically anti-isomorphic to A^{**} . Let ϕ be this anti-isomorphism. It is now easy to deduce from Corollary 2.2 that $\phi(I_A) = \pi(A)$.

Conversely, suppose that there exists an isometric anti-isomorphism of $M(A)$ onto A^{**} such that $\phi(I_A) = \pi(A)$. Since I_A is a closed left ideal

of $M(A)$, it follows that $\pi(A)$ is a closed right ideal of A^{**} . But $\pi(A)$ is a $*$ -subalgebra of A^{**} . Hence $\pi(A)$ is a closed two-sided ideal of A^{**} . Therefore, by [8, p. 533, Theorem 5.1], A is dual. This completes the proof.

As an immediate consequence of the proof of Theorem 3.1, we have:

COROLLARY 3.2. *A B^* -algebra A is dual if and only if every multiplier on $\pi(A)$ is given by the restriction to $\pi(A)$ of the right multiplication operator T_a , for some $a \in A^{**}$.*

THEOREM 3.3. *Let A be a B^* -algebra with minimal left ideals. Let I be a minimal left ideal of A , $[I]$ the closed two-sided ideal generated by I . Then A is $*$ -isomorphic to $L(H)$, for some Hilbert space H , if and only if $M([I])$ is isometrically anti-isomorphic to A .*

PROOF. Suppose A is $*$ -isomorphic to $L(H)$. Then the closed two-sided ideal generated by any minimal left ideal I of A is $*$ -isomorphic to $LC(H)$ and, by Corollary 2.2, $M(LC(H))$ is isometrically anti-isomorphic to $L(H)$.

Conversely suppose that $M([I])$ is isometrically anti-isomorphic to A . Let B be the closure of the socle of A . Then B is a nonzero dual B^* -algebra and every minimal left ideal of A is also a minimal left ideal of B . Hence, by [2, p. 158, Theorem 5], $[I]$ is a minimal closed two-sided ideal of B and therefore is $*$ -isomorphic to $LC(H)$, for some Hilbert space H . Hence, by Lemma 2.1, A is isometrically isomorphic to $L(H)$. [6, p. 248, Corollary (4.8.19)] now completes the proof.

COROLLARY 3.4. *Let A be a B^* -algebra containing minimal left ideals. Let I be a minimal left ideal of A and $[I]$ the closed two-sided ideal generated by I . Then A is $*$ -isomorphic to $L(H)$, for some Hilbert space H if and only if A is $*$ -isomorphic to the second conjugate space of $[I]$ considered as a B^* -algebra with Arens product.*

PROOF. This follows from the proof above and [6, p. 248, Corollary (4.8.19)].

For another characterization of the algebra $L(H)$, see [9, p. 537, Theorem 8].

REMARK. We observe that the Hilbert space H in Theorem 3.3 as well as in Corollary 3.4 is essentially unique. For if $L(H_1)$ is $*$ -isomorphic to $L(H_2)$, then H_1 is isometrically isomorphic to H_2 . (See [9, p. 538].)

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OTTAWA, OTTAWA, CANADA

Current address (Malviya): Department of Mathematics, North Texas State University, Denton, Texas 76203