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MULTIPLIER OPERATORS ON B^*-ALGEBRAS[1]

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ABSTRACT. The purpose of this paper is to give a characterization of the dual B^*-algebra and the algebra of bounded linear operators on Hilbert space in terms of their multipliers.

1. All algebras and vector spaces under consideration are over the complex field C. If A is a Banach algebra, A^* will denote the first conjugate space and A^{**} the second conjugate space of A. For any Hilbert space H, $L(H)$ will denote the algebra of continuous linear operators on H.

Let A be a B^*-algebra. Following Máté [5], we call a bounded linear operator T mapping A into itself a multiplier if $T(xy)=x(Ty)$ for all x, $y\in A$. The set $M(A)$ of all multipliers on A is a Banach algebra. For every $a\in A$, the right multiplication operator T_a is a multiplier on A, $I_A=\{T_a : a\in A\}$ is a closed left ideal of $M(A)$ and the mapping $a\to T_a$ is an isometric anti-isomorphism of A onto I_A.

From [3, p. 869, Theorem 7.1] it follows that if A is a B^*-algebra then the two Arens products defined on A^{**} coincide. For later use we sketch one of the Arens products. We do this in stages as follows [1], [3]: Let x, $y\in A, f\in A^*$, F, $G\in A^{**}$.

(i) Define $f\circ x$ by $(f\circ x)y=f(xy)$; $f\circ x\in A^*$.

(ii) Define $G\circ f$ by $(G\circ f)x=G(f\circ x)$; $G\circ f\in A^*$.

(iii) Define $F\circ G$ by $(F\circ G)f=F(G\circ f)$; $F\circ G\in A^{**}$.

For any Hilbert space H, $LC(H)$ will denote the subalgebra of $L(H)$ consisting of compact operators and $\tau c(H)$ the subalgebra of trace class operators on H. We shall denote the trace function of $\tau c(H)$ by $\mathrm{tr}(\cdot)$ and the trace norm by $\tau(\cdot)$; $\tau(T)=\mathrm{tr}((T^*T)^{1/2})$ for all $T\in\tau c(H)$. As a Banach space $\tau c(H)$ can be identified with the conjugate space of $LC(H)$ in the following way: For each continuous linear functional f on $LC(H)$ there exists a unique T in $\tau c(H)$ such that $f(S)=\mathrm{tr}(ST)$ for $S\in LC(H)$ and $\|f\|=\tau(T)$ [7, p. 46, Theorem 1]. Similarly, the conjugate space of $\tau c(H)$ can be identified (isometrically) with $L(H)$ [7, p. 47, Theorem 2].

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Thus the second conjugate space of $LC(H)$ is isometrically isomorphic to $L(H)$. In fact, it can be shown that this isomorphism is actually a *-isomorphism when the second conjugate space of $LC(H)$ is given the Arens product.

Let $\{A_\lambda : \lambda \in \Lambda\}$ be a family of Banach algebras. Let $\sum A_\lambda$ be the set of all functions on Λ with $f(x) \in A_\lambda$, for each λ, and such that $\|f\| = \sup_\lambda \|f(\lambda)\| < \infty$. Then under the usual operations for functions and the norm $\|f\|$, $\sum A_\lambda$ is a Banach algebra. It is called the normed full direct sum of the algebras A_λ [**6**, p. 77]. Let $(\sum A_\lambda)_0$ be the subset of $\sum A_\lambda$ consisting of all f such that, for every $\varepsilon > 0$, the set $\{\lambda : \|f(\lambda)\| \geqq \varepsilon\}$ is finite. Then $(\sum A_\lambda)_0$ is a closed subalgebra of $\sum A_\lambda$ [**6**, p. 107].

For any set S in a Banach algebra A, let $l(S)$ and $r(S)$ be the left and right annihilators of S, respectively. A is called dual if $l(r(J)) = J$ and $r(l(R)) = R$ for every closed left ideal J and every closed right ideal R of A. As usual $\mathrm{cl}(S)$ will denote the closure of S in A.

2. We devote this section to several lemmas which will be useful to us in §3.

LEMMA 2.1. *To each multiplier T on the algebra $LC(H)$ there corresponds a unique element a_T in $L(H)$ such that $T(s) = sa_T$ for all $s \in LC(H)$; $\|T\| = \|a_T\|$. Thus the mapping $T \to a_T$ is an isometric anti-isomorphism of $M(LC(H))$ onto $L(H)$.*

PROOF. Let $A = LC(H)$ and let $T \in M(A)$. Since A has an approximate identity, by [**5**, p. 810, Theorem 1], there exists a unique element F in A^{**} such that

$$(1) \qquad (F \circ f)s = f(T(s)) \qquad (s \in A, f \in A^*).$$

For $s \in A$ and $f \in A^*$, let t_f and $t_{f\circ s}$ be the elements of $\tau c(H)$ such that $f(a) = \mathrm{tr}(at_f)$ and $(f \circ s)(a) = \mathrm{tr}(at_{f\circ s})$ for all $a \in A$. (See [**7**, p. 46, Theorem 1].) Since

$$\mathrm{tr}(t_f sa) = f(sa) = (f \circ s)a = \mathrm{tr}(t_{f\circ s}a) \qquad (a \in A),$$

[**7**, p. 45, Lemma 1] shows that

$$(2) \qquad t_{f\circ s} = t_f s \qquad (s \in A, f \in A^*).$$

Thus

$$(3) \quad (F \circ f)s = F(f \circ s) = \mathrm{tr}(t_{f\circ s} t_F) = \mathrm{tr}(t_f s t_F) \qquad (s \in A, f \in A^*),$$

where t_F is the unique element in $L(H)$ such that $F(f) = \mathrm{tr}(t_f t_F)$ for all $f \in A^*$. (See [**7**, p. 47, Theorem 2].) But $f(T(s)) = \mathrm{tr}(t_f T(s))$. Hence from (1) and (3) it follows that

$$f(T(s)) = \mathrm{tr}(t_f s t_F) \qquad (f \in A^*).$$

Recalling [7, p. 45, Lemma 1], we see that $T(s)=st_F$ for all $s\in A$. Taking $a_T=t_F$ completes the proof.

COROLLARY 2.2. *Let $A=LC(H)$. Then there exists an isometric anti-isomorphism ϕ of $M(A)$ onto A^{**} such that $\phi(I_A)=\pi(A)$, where $\pi(A)$ is the canonical image of A in A^{**}.*

PROOF. For each $a\in A$, the right multiplication operator T_a is a multiplier on A. Hence by Lemma 2.1, there exists $b\in L(H)$ such that $T_a x=xb$ for all $x\in A$. But this means that $xa=xb$, for all $x\in A$, which clearly implies that $a=b$. Let ϕ_1 be the mapping $T\to a_T$ of $M(A)$ onto $L(H)$ given in Lemma 2.1, and let ϕ_2 be the mapping $a\to F_a$ which identifies $L(H)$ with A^{**}; ϕ_2 is an isometric *-isomorphism of $L(H)$ onto A^{**}. Let ϕ be the composite map $\phi=\phi_2\circ\phi_1$. Then ϕ is an isometric anti-isomorphism of $M(A)$ onto A^{**} such that $\phi(I_A)=\pi(A)$.

The following lemma is easy to prove and we state it mainly for convenience.

LEMMA 2.3. *Let $\{A_\lambda:\lambda\in\Lambda\}$ be a family of semisimple Banach algebras and let $A=(\sum A_\lambda)_0$. For each $\lambda\in\Lambda$, let $I_\lambda=\{f\in A:f(\mu)=0$ if $\mu\neq\lambda\}$ and $B_\lambda=\{f\in A:f(\lambda)=0\}$. Then*

(i) *$I_\lambda\cap B_\lambda=(0)$ and $I_\lambda+B_\lambda=A$.*

(ii) *$l(I_\lambda)=r(I_\lambda)=B_\lambda$ and $l(B_\lambda)=r(B_\lambda)=I_\lambda$.*

LEMMA 2.4. *Let A, I_λ, and B_λ be as in Lemma 2.3. Let $T\in M(A)$. Then*

(i) *T leaves each I_λ invariant, i.e., $T(I_\lambda)\subset I_\lambda$.*

(ii) *If T_λ denotes the restriction of T to I_λ, then*

$$\|T\| = \sup_\lambda \|T_\lambda\|.$$

PROOF. (i) Let $x\in B_\lambda$ and $y\in I_\lambda$. Then $0=T(xy)=xTy$ which shows that $Ty\in r(B_\lambda)=I_\lambda$ by Lemma 2.3. Hence $T(I_\lambda)\subset I_\lambda$.

(ii) Clearly $\|T_\lambda\|\leqq\|T\|$ for all λ. Let $\varepsilon>0$ be given. Then there exists $f\in A$, $\|f\|=1$, such that $\|T\|-\varepsilon\leqq\|Tf\|$. Since $A=(\sum A_\lambda)_0$, there exists $\lambda_1, \lambda_2, \cdots, \lambda_n$ such that $\|f(\lambda_i)\|\geqq\varepsilon/\|T\|$ and $\|f(\lambda)\|<\varepsilon/\|T\|$ for $\lambda\neq\lambda_i$ $(i=1,\cdots,n)$. Let $g\in A$ be such that $g(\lambda_i)=f(\lambda_i)$ and $g(\lambda)=0$ for $\lambda\neq\lambda_i$ $(i=1,2,\cdots,n)$. Then $\|f\|=\|g\|$ and

$$\|Tf\| = \|Tg\| = \sup_{1\leqq i\leqq n} \|T_{\lambda_i}(g(\lambda_i))\|,$$

so that $\|Tf\|=\|T_{\lambda_{i_0}}(g(\lambda_{i_0}))\|$ for some i_0, $1\leqq i_0\leqq n$. Since $\|T_{\lambda_{i_0}}(g(\lambda_{i_0}))\|\leqq\|T_{\lambda_{i_0}}\|$, we have $\|T\|-\varepsilon\leqq\|T_{\lambda_{i_0}}\|$. Hence $\|T\|=\sup_\lambda\|T_\lambda\|$.

LEMMA 2.5. *Let $\{A_\lambda:\lambda\in\Lambda\}$ be a family of semisimple Banach algebras and let $A=(\sum A_\lambda)_0$. Then $M(A)$ is isometrically isomorphic to the normed full direct sum of the algebras $M(A_\lambda)$.*

PROOF. For each $\lambda\in\Lambda$, let $I_\lambda=\{f\in A: f(\mu)=0$ if $\mu\neq\lambda\}$ and, for each $T\in M(A)$, let T_λ be the restriction of T to I_λ; T_λ is a multiplier on I_λ. Since A_λ is isometrically isomorphic to I_λ, each T_λ may be identified as an element of $M(A_\lambda)$ with the same norm. For $T\in M(A)$, let $\mathscr{T}_T$ be the function on Λ such that $\mathscr{T}_T(\lambda)=T_\lambda$. By Lemma 2.4, $\mathscr{T}_T$ is an element of the normed full direct sum $\sum M(A_\lambda)$ with $\|\mathscr{T}_T\|=\|T\|$. Hence $T\to\mathscr{T}_T$ is an isometric isomorphism of $M(A)$ into $\sum M(A_\lambda)$. To show that this mapping is onto, let $\mathscr{T}\in\sum M(A_\lambda)$ and let T be the mapping on A such that $(Tf)(\lambda)=\mathscr{T}(\lambda)f(\lambda)$. It is easy to see that T is a multiplier on A with $\|T\|=\|\mathscr{T}\|$. Thus $T\to\mathscr{T}_T$ is onto and this completes the proof.

COROLLARY 2.6. *Let A be a dual B^*-algebra and let $\{I_\lambda:\lambda\in\Lambda\}$ be the family of all minimal closed two-sided ideals of A. For each $T\in M(A)$ and $\lambda\in\Lambda$, let T_λ be the restriction of T to I_λ. Let $M_\lambda=\{T_\lambda: T\in M(A)\}$. Then $M(A)$ is isometrically isomorphic to the normed full direct sum of the algebras M_λ.*

PROOF. By [**6**, p. 267, Theorem (4.10.14)], $A=(\sum I_\lambda)_0$ and so, by Lemma 2.5, $M(A)$ is isometrically isomorphic to the normed full direct sum $\sum M(I_\lambda)$. Now, since $I_\lambda\cap r(I_\lambda)=(0)$ and $I_\lambda+r(I_\lambda)=A$, it is easy to show that $M_\lambda=M(I_\lambda)$. Hence $M(A)$ is isometrically isomorphic to $\sum M_\lambda$.

3. We are now ready to prove the characterizations mentioned in the abstract.

THEOREM 3.1. *Let A be a B^*-algebra, A^{**} its second conjugate space and $\pi(A)$ the canonical image of A in A^{**}. Give A^{**} the Arens product. Then A is a dual algebra if and only if there exists an isometric anti-isomorphism ϕ of $M(A)$ onto A^{**} such that $\phi(I_A)=\pi(A)$.*

PROOF. Suppose that A is dual. Then there exists a family of Hilbert spaces $\{H_\lambda:\lambda\in\Lambda\}$ such that A is *-isomorphic to $(\sum LC(H_\lambda))_0$ [**4**, p. 221, Lemma 2.3]. It now follows that A^* is isometrically isomorphic to $(\sum \tau c(H_\lambda))_1$, the L_1-direct sum of the algebras $\tau c(H_\lambda)_1$, and that in turn A^{**} is isometrically isomorphic to the normed full direct sum $\sum L(H_\lambda)$ of the algebras $L(H_\lambda)$ [**8**, p. 532]. Letting $LC(H_\lambda)=A_\lambda$ and identifying A with $(\sum A_\lambda)_0$, Lemma 2.5 shows that $M(A)$ is isometrically isomorphic to the normed full direct sum of the algebras $M(A_\lambda)$. But, by Corollary 2.2, $M(A_\lambda)$ is isometrically anti-isomorphic to $L(H_\lambda)$, for each $\lambda\in\Lambda$. Hence $M(A)$ is isometrically anti-isomorphic to $\sum L(H_\lambda)$. Since $\sum L(H_\lambda)$ is *-isomorphic to A^{**}, it follows that $M(A)$ is isometrically anti-isomorphic to A^{**}. Let ϕ be this anti-isomorphism. It is now easy to deduce from Corollary 2.2 that $\phi(I_A)=\pi(A)$.

Conversely, suppose that there exists an isometric anti-isomorphism of $M(A)$ onto A^{**} such that $\phi(I_A)=\pi(A)$. Since I_A is a closed left ideal of $M(A)$, it follows that $\pi(A)$ is a closed right ideal of A^{**}. But $\pi(A)$ is a *-subalgebra of A^{**}. Hence $\pi(A)$ is a closed two-sided ideal of A^{**}. Therefore, by [**8**, p. 533, Theorem 5.1], A is dual. This completes the proof.

As an immediate consequence of the proof of Theorem 3.1, we have:

COROLLARY 3.2. *A B^*-algebra A is dual if and only if every multiplier on $\pi(A)$ is given by the restriction to $\pi(A)$ of the right multiplication operator T_a, for some $a \in A^{**}$.*

THEOREM 3.3. *Let A be a B^*-algebra with minimal left ideals. Let I be a minimal left ideal of A, $[I]$ the closed two-sided ideal generated by I. Then A is *-isomorphic to $L(H)$, for some Hilbert space H, if and only if $M([I])$ is isometrically anti-isomorphic to A.*

PROOF. Suppose A is *-isomorphic to $L(H)$. Then the closed two-sided ideal generated by any minimal left ideal I of A is *-isomorphic to $LC(H)$ and, by Corollary 2.2, $M(LC(H))$ is isometrically anti-isomorphic to $L(H)$.

Conversely suppose that $M([I])$ is isometrically anti-isomorphic to A. Let B be the closure of the socle of A. Then B is a nonzero dual B^*-algebra and every minimal left ideal of A is also a minimal left ideal of B. Hence, by [**2**, p. 158, Theorem 5], $[I]$ is a minimal closed two-sided ideal of B and therefore is *-isomorphic to $LC(H)$, for some Hilbert space H. Hence, by Lemma 2.1, A is isometrically isomorphic to $L(H)$. [**6**, p. 248, Corollary (4.8.19)] now completes the proof.

COROLLARY 3.4. *Let A be a B^*-algebra containing minimal left ideals. Let I be a minimal left ideal of A and $[I]$ the closed two-sided ideal generated by I. Then A is *-isomorphic to $L(H)$, for some Hilbert space H if and only if A is *-isomorphic to the second conjugate space of $[I]$ considered as a B^*-algebra with Arens product.*

PROOF. This follows from the proof above and [**6**, p. 248, Corollary (4.8.19)].

For another characterization of the algebra $L(H)$, see [**9**, p. 537, Theorem 8].

REMARK. We observe that the Hilbert space H in Theorem 3.3 as well as in Corollary 3.4 is essentially unique. For if $L(H_1)$ is *-isomorphic to $L(H_2)$, then H_1 is isometrically isomorphic to H_2. (See [**9**, p. 538].)

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