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CONVOLUTION OF $L(p, q)$ FUNCTIONS

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ABSTRACT. In the present paper, examples are given to show that the convolution theorem, which is the $L(p,q)$ analogue of Young's inequality for the L^p spaces, is best possible. This result is then used to obtain a theorem about bounded linear translation invariant operators between two $L(p,q)$ spaces.

Introduction. Let G be a noncompact, locally compact group and let $L^p=L^p(G)$, $1\leqq p\leqq\infty$, denote the Lebesgue spaces associated with G. N. W. Rickert [5] has proved that the convolution of two L^p functions need not exist. In this paper we extend this result to the $L(p,q)(G)$ spaces. For our purposes this is done first where the underlying group is one of the Euclidean n spaces R^n, $R^1=R$ and the convolution of two measurable functions f and g defined on R^n is ordinary convolution. That is,

$$f * g(y)=\int_{R^n} f(y-x)g(x)\,dx, \qquad y\in R^n.$$

This result is then used to obtain a theorem about bounded linear translation invariant operators between two $L(p,q)$ spaces.

For a discussion of the $L(p,q)$ spaces the reader is referred to R. A. Hunt [4].

1. Let R^n, $R^1=R$ be the Euclidean n spaces and let $L(p,q)=L(p,q)(R^n)$ be a Lorentz space for which $f\in L(p,q)$, implies f is locally integrable. Namely, we will consider only those $L(p,q)$ spaces for the particular choice of indices $1<p<\infty$, $0<q\leqq\infty$; $p=q=1$ or $p=q=\infty$. (The spaces $L(\infty,q)$, $0<q<\infty$, will be omitted since $f\in L(\infty,q)$, implies $f=0$ a.e.)

THEOREM 1. *If* $1<p_i\leqq\infty$, $0<q_i\leqq\infty$ $(i=1,2)$, *where*

(1) $1/p_1+1/p_2<1$ *or*

(2) $1/p_1+1/p_2=1$, $1/q_1+1/q_2<1$,

then there are functions $f\in L(p_1,q_1)$, $g\in L(p_2,q_2)$ *such that* $f*g(y)=\infty$ *for* y *in* R^n.

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PROOF. Assume $n=1$. We first prove (1). If $p_1<\infty$ or $p_2<\infty$, then choose $\alpha>0$ such that $\alpha(1/p_1+1/p_2)=1$ and define

$$\begin{aligned} f(x) &= 1, && \text{if } |x| \leqq 1, \\ &= |x|^{-\alpha/p_1}, && \text{if } |x| > 1. \end{aligned}$$

Define $g(x)$ similarly, only this time with respect to α and p_2. In the case $p_1=p_2=\infty$, define $f(x)=g(x)=1$ for x in R.

It is easy to show that the rearrangement functions f^* and g^* of the functions f and g are given by

$$\begin{aligned} f^*(t) &= f(t/2), && t > 0, \\ g^*(t) &= g(t/2), && t > 0. \end{aligned}$$

Then a calculation of $\|f\|^*_{(p_1,q_1)}$ and $\|g\|^*_{(p_2,q_2)}$, which is straightforward, shows that $f \in L(p_1, q_1)$ and $g \in L(p_2, q_2)$.

We next show that $f * g(y) = \infty$ for each y in R. By symmetry it is enough to assume $y \geqq 0$.

$$\begin{aligned} f * g(y) &= \int_{-\infty}^{\infty} f(y-x)g(x)\,dx \\ &\geqq \int_{1+y}^{\infty} (x-y)^{-\alpha/p_1} x^{-\alpha/p_2}\,dx \geqq \int_{1+y}^{\infty} x^{-1}\,dx = \infty. \end{aligned}$$

To prove (2), choose α, β such that $0<\alpha<q_1$, $0<\beta<q_2$, $0 \leqq 1/(q_1-\alpha)+1/(q_2-\beta)<1$ and define

$$\begin{aligned} f(x) &= 2^{-1/p_1}(\log 2)^{1/(\alpha-q_1)}, && \text{if } |x| \leqq 2, \\ &= |x|^{-1/p_1}(\log|x|)^{1/(\alpha-q_1)}, && \text{if } |x| > 2. \end{aligned}$$

Define $g(x)$ similarly, only this time with respect to β, p_2, q_2 and proceed as in the proof of part (1).

Finally, assume $n>1$. Let h defined on R^{n-1} be a function of rapid descent (see [3, p. 97]) such that $h(y)>0$, for y in R^{n-1}. If the indices p_1, p_2, q_1, q_2 are as in part (1) or (2), put $f_1(x_1)=f(x_1)$, where f is as before and define $f(x)=f_1(x_1)h(x_2, \cdots, x_n)$, $x \in R^n$. Define $g(x)$ similarly. The rest of the proof now becomes straightforward. We omit the details.

We remark that part (1) of Theorem 1 is an adaptation of a theorem of N. W. Rickert [5] about L^p, $1<p\leqq\infty$, spaces. Based on Rickert's theorem for the L^p spaces, Theorem 1 has the following generalization: *If the indices* p_1, p_2, q_1, q_2 *are as in Theorem* 1 *and G is a noncompact, locally compact group, then there is an open set U in G and there are functions* $f \in L(p_1, q_1)(G)$, $g \in L(p_2, q_2)(G)$ *such that* $f * g(y)$ *is not defined for y in U.*

REMARK. Theorem 1 has a converse. It can be shown that (see **[1]**): If $1 \leqq p_i \leqq \infty$, $0<q_i\leqq\infty$ $(i=1,2)$, where (1) $1/p_1+1/p_2>1$ or (2) $1/p_1+1/p_2=1$, $1/q_1+1/q_2\geqq 1$, then $f \in L(p_1, q_1)$ and $g \in L(p_2, q_2)$ implies $f*g \in L(r, s)$

where $1/r=1/p_1+1/p_2-1$, $1/s\leqq 1/q_1+1/q_2$. Moreover,

$$\|f * g\|^*_{(r,s)} \leqq B\ \|f\|^*_{(p_1,q_1)}\ \|g\|^*_{(p_2,q_2)},$$

where $B>0$ is a constant which depends only on the indices r, s, p_i, q_i $(i=1, 2)$.

2. Let $L(p,q)=L(p,q)(R^n)$, where R^n, $R^1=R$ is one of the Euclidean n spaces. If $h\in R^n$, denote by τ_h the operator defined by $(\tau_h f)(x)=f(x+h)$, $x\in R^n$, for each measurable function f defined on R^n. A bounded linear operator T from $L(p_1,q_1)$ to $L(p_2,q_2)$ is translation invariant if $\tau_h T=T\tau_h$, $h\in R^n$. Such operators between the classical L^p, $1\leqq p\leqq\infty$, spaces were studied by L. Hörmander [**3**].

Let $L_0(p,q)$ be the subspace of $L(p,q)$ obtained by taking the closure in $L(p,q)$, with respect to the metric topology of $L(p,q)$, of the set of simple functions having compact support. It can be shown that (see [**1**]):

(1) If $1<p<\infty$, $0<q<\infty$ or $p=q=1$, then $L_0(p,q)=L(p,q)$; (2) If $p=q=\infty$, then $L_0(\infty, \infty)=L_0^\infty$ is the subspace of functions in L^∞ which tend to 0 at infinity. (3) If $1<p<\infty$, $q=\infty$, then $L_0(p, \infty)$ is the subspace of $L(p, \infty)$ consisting of those functions for which $t^{1/p}f^*(t)$ converges to 0 as t tends to 0^+ and ∞.

L. Hörmander [**3**, p. 96] showed that nontrivial translation-invariant operators need not exist between two L^p spaces. Using the definition of a rearrangement function f^* of a function f and Hörmander's proof, this result extends easily to the $L(p,q)$ spaces. And takes the form: *If T is a bounded linear translation invariant operator from $L(p_1,q_1)$ to $L_0(p_2,q_2)$, such that $p_1>p_2$, then $T=0$ when restricted to $L_0(p_1,q_1)$.*

Our purpose here is to prove the following theorem.

THEOREM 2. *If T is a bounded linear translation invariant operator from $L(p,q_1)$ to $L(p,q_2)$, $1<p<\infty$, such that*

(1) $f\geqq 0$, implies $Tf\geqq 0$,

(2) $q_1>q_2$,

then $T=0$ if $q_1<\infty$ and if $q_1=\infty$ then the restriction of T to $L_0(p, \infty)$ is 0.

PROOF. If T is a bounded linear translation invariant operator from $L(p_1,q_1)$ to $L(p_2,q_2)$, then T has the characterization (see [**1**]) that: if $g\in L(1, 1)$ and $f\in L_0(p_1,q_1)$, then $[T(f * g)](y)=[g * Tf](y)$, $y\in R^n$. This will be used later in the proof.

The continuity of T implies the existence of a constant $B\geqq 0$ such that

$$\|Tf\|^*_{(p,q_2)} \leqq B\ \|f\|^*_{(p,q_1)}, \qquad f\in L(p, q_1).$$

If $B=0$, then $T=0$. Assume $B>0$ and choose indices r, s such that $1/p+1/r=1$, $1/s+1/q_2=1$. Since $q_1>q_2$, it follows that $1/s+1/q_1<1$. Let

$f\in L(p, q_1)$ and $g\in L(r, s)$ be as in Theorem 1. Let

$$E_n = \{x: 1/n < f(x) \leqq n\} \cap \{x: |x| \leqq n\},$$
$$F_n = \{x: 1/n < g(x) \leqq n\} \cap \{x: |x| \leqq n\}, \qquad n = 1, 2, \cdots.$$

Put $f_n(x)=f(x)\chi_{E_n}(x)$ and $g_n(x)=g(x)\chi_{F_n}(x)$, where χ_E denotes the characteristic function of the measurable set E. It follows that $f_n\in L(1, 1)\cap L_0(p, q_1)$ and $g_n\in L(1, 1)\cap L_0(r, s)$, $n=1, 2, \cdots$. Moreover, $f_n\uparrow f$ and $g_n\uparrow g$ pointwise everywhere as n tends to infinity.

Next, let $u(x)$ defined on R^n be a nonnegative simple function having compact support. Using the remarks at the end of §1 we have

$$\begin{aligned} B \|u\|^*_{(1,1)} \|f\|^*_{(p,q_1)} \|g\|^*_{(r,s)} & \\ &\geqq B \|u\|^*_{(1,1)} \|f_n\|^*_{(p,q_1)} \|g_n\|^*_{(r,s)} \geqq \|u\|^*_{(1,1)} \|Tf_n\|^*_{(p,q_2)} \|g_n\|^*_{(r,s)} \\ &\geqq C \|u * Tf_n\|^*_{(p,q_2)} \|g_n\|^*_{(r,s)} \geqq C \|u * Tf_n * g_n\|^*_{(\infty,\infty)} \\ &= C \|f_n * g_n * Tu\|^*_{(\infty,\infty)} \geqq C(f_n * g_n * Tu)(y), \end{aligned}$$

$y\in R^n$, and a constant $C>0$ which depends only on the indices p, q_2, r, s, $1, q_1$. By condition (1) and the monotone convergence theorem, it follows that

$$B \|u\|^*_{(1,1)} \|f\|^*_{(p,q_1)} \|g\|^*_{(r,s)} \geqq C(f * g * Tu)(y), \qquad y \in R^n.$$

By the way the functions f and g were constructed, the term on the right equals ∞ for each y in R^n, unless $Tu(x)=0$ a.e. $x\in R^n$.

If u is an arbitrary simple function having compact support, then $u=u^+-u^-$, where u^+ and u^- are nonnegative simple functions having compact support. By the linearity of the operator T we have $Tu=Tu^+-Tu^-=0$. This implies $T=0$ on $L_0(p, q_1)$.

REMARK. If for the choice of indices p_1, p_2, q_1, q_2 either $p_1<p_2$ or $p_1=p_2$, $q_1\leqq q_2$, then by using the remark at the end of §1 it is easy to construct a nontrivial bounded linear translation invariant operator from $L(p_1, q_1)$ to $L(p_2, q_2)$.

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