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AN EXAMPLE IN THE THEORY OF WELL-BOUNDED OPERATORS

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ABSTRACT. If H is the Hilbert transform on $L^p(\boldsymbol{Z})$, then $T=\pi I+iH$ is a well-bounded operator for $1<p<\infty$, but is not a scalar-type spectral operator except when $p=2$.

The purpose of this note is to show that there is a well-bounded operator on a reflexive Banach space which is not scalar-type spectral.

Throughout, let X be a reflexive complex Banach space, and T a bounded linear operator on X. T is said to be well-bounded if and only if it is possible to choose a constant K and a compact interval $J=[a, b]$, such that

$$\|f(T)\| \leqq K\left\{\sup_{\lambda\in J} |f(\lambda)| + \operatorname*{var}_J f\right\} \tag{1}$$

for every complex polynomial f. Such an operator has real spectrum, contained in J. Let $\boldsymbol{R}$ denote the set of real numbers. Smart [5, pp. 329–330] and Ringrose [4] have shown that T is well-bounded and satisfies (1) if and only if there is a family $\{E(\lambda):\lambda\in\boldsymbol{R}\}$ of projections on X such that

(i) $\|E(\lambda)\|\leqq K$, $\lambda\in\boldsymbol{R}$;
(ii) $\lim_{\lambda\to\mu+} E(\lambda)x=E(\mu)x$, $x\in X$;
(iii) $\lim_{\lambda\to\mu-} E(\lambda)$ exists in the strong operator topology;
(iv) $E(\lambda)=0$, $\lambda<a$; $E(\lambda)=I$, $\lambda\geqq b$;
(v) $T=\int_J \lambda\, dE(\lambda)$,

where the integral exists as a strong limit of Riemann sums.

Now let S be a scalar-type spectral operator on X with real spectrum $\sigma(S)$, contained in the compact interval J. It is well known that there is a constant M such that

$$\|f(S)\| \leqq M \sup_{\lambda\in\sigma(S)} |f(\lambda)| \leqq M \sup_{\lambda\in J} |f(\lambda)| \tag{2}$$

for every complex polynomial f. Moreover, this property characterises the scalar-type spectral operators on X with real spectra [1, Theorem 5.3].

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Hence every such operator is well-bounded. We show that the converse result fails by proving that the Hilbert transform on $L^p(\boldsymbol{Z})$ $(1<p<\infty)$ determines a well-bounded operator which is not scalar-type spectral unless $p=2$.

Let $\boldsymbol{Z}$ be the locally compact abelian group of integers with (Haar) counting measure. Let $\boldsymbol{Z}^\wedge$ be its dual group, the circle, which is isomorphic to the interval $[0, 2\pi]$ with its endpoints identified. Haar measure on $[0, 2\pi]$ is Lebesgue measure divided by 2π. We denote the norm on $L^p(\boldsymbol{Z})$ by $\|\cdot\|_p$.

Let $\tau: L^2(\boldsymbol{Z}^\wedge)\to L^2(\boldsymbol{Z})$ be the Plancherel extension of the Fourier transform, and let $^\wedge$ be the inverse of τ. If $x, y\in L^2(\boldsymbol{Z}^\wedge)$, then $\tau(xy)=\tau(x)*\tau(y)$, where $*$ denotes convolution; also

$$(3)\qquad (\tau x)(n) = \frac{1}{2\pi}\int_0^{2\pi} x(\lambda)e^{-in\lambda}\,d\lambda \qquad (n\in \boldsymbol{Z},\ x\in L(\boldsymbol{Z}^\wedge)).$$

Let $e_\theta=\tau\chi_{[0,\theta]}$ $(0\leqq\theta\leqq 2\pi)$. Then $e_\theta(0)=\theta/2\pi$ and $e_\theta(n)=(1-e^{-in\theta})/2\pi ni$ $(n\neq 0)$. Thus $e_\theta\in L^p(\boldsymbol{Z})$ $(p>1)$ and the set $\{e_\theta: 0\leqq\theta\leqq 2\pi\}$ is dominated in $L^p(\boldsymbol{Z})$ $(p>1)$. The identity of the group algebra $L^1(\boldsymbol{Z})$ is $e_{2\pi}$.

Let j be the function $j:[0, 2\pi]\to[0, 2\pi]:\lambda\mapsto\lambda$ and let $\delta=\tau j$. Then $\delta(0)=\pi$ and $\delta(n)=i/n$ $(n\neq 0)$. The Hilbert transform H is defined on $L^p(\boldsymbol{Z})$ $(p>1)$ by

$$(4)\quad H\xi = \xi * (\cdots, -1/n, \cdots, -1/2, -1, 0, 1, 1/2, \cdots, 1/n, \cdots).$$

Stečkin [6, Corollary 2] has shown that $E(\theta):\xi\mapsto\xi * e_\theta$ $(0\leqq\theta\leqq 2\pi)$ defines a bounded linear operator on $L^p(\boldsymbol{Z})$ $(p>1)$ and that

$$\sup\{\|E(\theta)\|: \theta\in[0, 2\pi]\} = K < \infty.$$

Now let p be fixed in the range $(1, 2)$. Let $0\leqq\theta\leqq\varphi\leqq 2\pi$, and let $\xi\in L^p(\boldsymbol{Z})\subset L^2(\boldsymbol{Z})$. Then

$$\begin{aligned}(\xi * e_\theta) * e_\varphi &= \tau(\hat{\xi}\chi_{[0,\theta]}) * \tau\chi_{[0,\varphi]} = \tau(\hat{\xi}\chi_{[0,\theta]}\chi_{[0,\varphi]})\\ &= \xi * e_\theta = (\xi * e_\varphi) * e_\theta.\end{aligned}$$

Hence $\{E(\theta): 0\leqq\theta\leqq 2\pi\}$ is a naturally ordered uniformly bounded family of projections on $L^p(\boldsymbol{Z})$.

Since $\|e_\theta-e_\varphi\|_2=(|\theta-\varphi|/2\pi)^{1/2}$, we have $\lim_{\varphi\to\theta} e_\varphi(n)=e_\theta(n)$ $(n\in\boldsymbol{Z})$. By the Lebesgue dominated convergence theorem, $\lim_{\varphi\to\theta}\|e_\varphi-e_\theta\|_p=0$. For ξ in $L^p(\boldsymbol{Z})$ and $n\geqq 1$ define

$$\xi_n = (\cdots, 0, \xi(-n), \xi(-n+1), \cdots, \xi(n-1), \xi(n), 0, \cdots).$$

Then

$$\begin{aligned}\|\xi * (e_\theta - e_\varphi)\|_p &\leqq \|(\xi-\xi_n) * (e_\theta - e_\varphi)\|_p + \|\xi_n * (e_\theta - e_\varphi)\|_p\\ &\leqq 2K\,\|\xi-\xi_n\|_p + \|\xi_n\|_1\,\|e_\theta - e_\varphi\|_p.\end{aligned}$$

Hence if $0<\theta<2\pi$,

$$E(\theta+) = \lim_{\varphi\to\theta+} E(\varphi) = E(\theta),$$
$$E(\theta-) = \lim_{\varphi\to\theta-} E(\varphi) = E(\theta),$$
$$E(0+) = E(0) = 0, \qquad E(2\pi-) = E(2\pi) = I,$$

the limits being defined in the strong operator topology. Therefore $T=\int_{[0,2\pi]} \lambda\, dE(\lambda)$ is a well-bounded operator.

Let $\{\lambda_k : k=0, 1, \cdots, m\}$ be a partition of $[0, 2\pi]$, and let f be any complex polynomial. Then, for every ξ in $L^p(\boldsymbol{Z})$,

$$\sum_{k=1}^{m} f(\lambda_k)[E(\lambda_k) - E(\lambda_{k-1})]\xi = \tau\left\{\sum_{k=1}^{m} f(\lambda_k)[\chi_{[0,\lambda_k]} - \chi_{[0,\lambda_{k-1}]}]\hat{\xi}\right\}.$$

Therefore

$$f(T)\xi = \tau(f\hat{\xi}). \tag{5}$$

In particular, $T\xi=\tau(j\hat{\xi})=\delta*\xi$; thus $T=\pi I+iH$.

Now suppose that T is a scalar-type spectral operator, and let g be any complex function continuous on $[0, 2\pi]$. From (3) and (5) we obtain

$$(f(T)e_{2\pi})(n) = \frac{1}{2\pi}\int_0^{2\pi} f(\lambda)e^{-in\lambda}\, d\lambda \tag{6}$$

for every complex polynomial f. Also $g(T)$ is a bounded linear operator on $L^p(\boldsymbol{Z})$ and from (2), (6) and the Weierstrass polynomial approximation theorem

$$(g(T)e_{2\pi})(n) = \frac{1}{2\pi}\int_0^{2\pi} g(\lambda)e^{-in\lambda}\, d\lambda.$$

This gives a contradiction, since not every complex function continuous on $[0, 2\pi]$ has $L^p(\boldsymbol{Z})$-summable Fourier coefficients [**7**, V. 4.11]. We have therefore shown that if $1<p<2$, then T is well-bounded but not scalar-type spectral. Now consider the case $2<p<\infty$, and let $1/p+1/q=1$. Then $1<q<2$, and from (4), H^* is the Hilbert transform on $L^q(\boldsymbol{Z})$. Therefore T^* is well-bounded but not scalar-type spectral. From the characterisations (1) and (2), it follows that T is well-bounded but not scalar-type spectral. We observe at this point that T is not even spectral if $1<p<2$ or $2<p<\infty$ by [**1**, Theorem 5.8]. Finally, if $p=2$, the projections $\{E(\theta):\theta\in[0, 2\pi]\}$ are easily seen to be selfadjoint and so T is selfadjoint. Hence in this case T is both scalar-type spectral and well-bounded.

The operator T has also been discussed in [**2**, pp. 452–461] and [**3**, p. 186].

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