JACOBI'S BOUND FOR FIRST ORDER DIFFERENCE EQUATIONS

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Abstract. Let $A_1, \ldots, A_n$ be a system of difference polynomials in $y^{(1)}, \ldots, y^{(n)}$, and let $\mathcal{M}$ be an irreducible component of the difference variety $\mathcal{M}(A_1, \ldots, A_n)$. If $r_i$ is the order of $A_i$ in $y^{(i)}$, the Jacobi number $J$ of the system is defined to be

$$
\max\{\sum_{i=1}^n r_{ij} j_{i1}, \ldots, j_{in} \text{ is a permutation of } 1, \ldots, n\}.
$$

In this paper it is shown for first order systems that if $\dim \mathcal{M} = 0$, then $E \text{ord. } \mathcal{M} \leq J$. The methods used are analogous to those used to obtain the corresponding result for differential equations (given in a recent paper by the author).

1. Introduction. For any system of difference polynomials $A_1, \ldots, A_n$, with $r_{ij}$ the order of $A_i$ in $y^{(j)}$, the Ritt number

$$
R = \sum_{j=1}^n \max\{r_{ij} : i = 1, \ldots, n\}
$$

provides a bound for the effective order of an irreducible component $\mathcal{M}$ of dimension 0 [1, pp. 253–255]. Greenspan improved this bound for those systems in which every irreducible component has dimension 0. (See [1, pp. 256–258].) In the case $n=2$, the Greenspan number coincides with the Jacobi number $J$. The Jacobi number can also be verified for linear difference systems by a proof analogous to that for the differential case [3].

The definitions of [1] are assumed, and the notation of [2] will be used.

2. Specialization problem. Let $K$ be an inversive difference field with automorphism $\tau$. Let $R$ be a difference kernel with principal realization $\bar{\alpha}$. In general, if $R$ is a kernel which specializes to $\bar{R}$, there may be no principal realization of $R$ which specializes to $\bar{\alpha}$. (See [1, p. 322, Example 2].) The following theorem presents in one case which such a specialization of principal realizations does exist.
THEOREM 1. Let $K$ be inversive. Let $R$ and $\bar{R}$ be difference kernels consisting of $K(a_n, \ldots, a_0)$, $T$, and $K(\bar{a}_n, \ldots, \bar{a}_0)$, $\bar{T}$, respectively, $r \geq 0$, with $K(a_n, \ldots, a_{r-1}) \cong_K K(\bar{a}_n, \ldots, \bar{a}_{r-1})$. Let $\bar{a}$ be a principal realization of $\bar{R}$. If $R$ specializes to $\bar{R}$, then there exists a principal realization $\alpha$ of $R$ which specializes to $\bar{a}$.

PROOF. It may be assumed that the kernels are of length 0 or 1 [1, p. 160]. The proof is given for kernels of length 1 (the length 0 case is similar). By an argument like that in Theorem 1 of [2], we may assume that $a = \bar{a}$, deg $\bar{R} = 0$ and deg $R = 1$.

It will be sufficient to show that for any $h > 1$ a kernel $K(a_n, \ldots, a_0)$, $T$, can be found with $K(a_n, \ldots, a_{k+1})$ a generic prolongation of $K(a_n, \ldots, a_0)$, $k = 1, \ldots, h-1$, such that $(a_n, \ldots, a_0)$ specializes to $(\bar{a}_n, \bar{a}_2, \ldots, \bar{a}_h)$, where $\bar{a}$ is the principal realization of $\bar{R}$. For suppose the theorem is false: then no principal realization of $R$ specializes to $\bar{a}$. Since the number of distinct principal realizations is finite [1, p. 156], there exists an integer $H > 1$ such that for any principal realization $\alpha$ of $R$, $(\alpha, \ldots, \alpha_H)$ does not specialize to $(\bar{a}, \ldots, \bar{a}_H)$.

Let $L$ be the algebraic closure of $K(\bar{a})$. By taking free joins and isomorphic kernels if necessary, we may assume that $(\bar{a}, \bar{a}_1) = (a, a_1)$, that t.d. $L(a_1)/L = 1$, and that $(a, a_1) \overset{T}{\sim} (a, \bar{a}_1)$. Then for any given $h > 1$, there exist vectors $b_2, \ldots, b_h$ such that

(i) $K(a_n, b_2, \ldots, b_h) \cong_K K(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_h)$,
(ii) $(a, a_1, b_2, \ldots, b_h) \overset{T}{\sim} (a, \bar{a}_1, b_2, \ldots, \bar{a}_h)$,
(iii) t.d. $L(a, a_1, b_2, \ldots, b_h)/L = 1$

(by Lemma 1 of [2] and the isomorphism $K(a_1) \cong_K K(\bar{a}_1)$).

Since (ii) and (iii) hold, Lemma 2 of [2] may be applied to obtain parameter $t \in L(a_1, b_2, \ldots, b_h)$, transcendental over $L$, such that $L[a_1, b_2, \ldots, b_h] \subseteq L[[t]]$ with

$$a_1^{(i)} = a_2^{(i)} + \sum_{j=1}^{\infty} c_{ij} t^j, \quad i = 1, \ldots, n;$$

$$b_k^{(i)} = \bar{a}_k^{(i)} + \sum_{j=1}^{\infty} d_{kj} t^j, \quad i = 1, \ldots, n; \quad k = 2, \ldots, h.$$  

The transformation of the kernel $R$, $T: K(a) \to K(a_1)$, may be extended to an isomorphism $T$ of $K(a, a_1, a_2, \ldots, a_{h-1})$ onto $K(a_1, b_2, \ldots, b_h)$, obtained by the composition of $T$ and the isomorphism of (i):

$$T: K(a, \bar{a}_1, \ldots, a_{h-1}) \xrightarrow{T} K(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_h) \xrightarrow{K} K(a_1, b_2, \ldots, b_h).$$
T may then be extended to a monomorphism of $L$ into a field $L_1$, which is the algebraic closure of the quotient field of $L[[r]]$. Then $T$ extends to a monomorphism of $L[[t]]$ into $L_1[[t]]$, where $t_1$ is transcendental over $L_1$ and $T(t)=t_1$. Repeating this method of extension, for each $k=2, \ldots, h-1$, one may extend $T$ to a monomorphism $T: L_{k-1}[[t_{k-1}]] \to L_k[[t_k]]$, with $L_{k-1}[[t_{k-1}]] \subseteq L_k$ and $t_k$ transcendental over $L_k$.

Since $a_t \in L[[t]]$, $T^k a = T^{k-1} a_t \in L_{k-2}[[t_{k-2}]]$, $k = 2, \ldots, h$. Let $a_k = T^k a$. Then $K[a, a_1, \ldots, a_{h-1}] \subseteq L_{h-2}[[t_{h-2}]]$, and $T$ restricted to $K[a, a_1, \ldots, a_h]$ is an isomorphism onto $K[a_1, \ldots, a_h]$. Thus $K(a, a_1, \ldots, a_h)$ is a kernel which can be obtained from $R$ by $h-1$ prolongations.

In the following let $b$ denote the vector $(b_2, \ldots, b_h)$. It will be shown that $K(a, a_1, \ldots, a_h)$ is obtained by generic prolongations. Since $t \in L(a, b)$, $t_k = T^k t \in L_k(a_{k+1}, T^k b)$, $1 \leq k \leq h-1$. But $t_k$ is transcendental over $L_k$; thus

(2) \[ \text{t.d. } L_k(a_{k+1}, T^k b)/L_k \geq 1. \]

It may be noted that

(3) \[ \text{t.d. } K(a, a_1, \ldots, a_k)/K(a, a_1, \ldots, a_k) \leq \text{t.d. } K(a_{k+1}, T^k b)/K(a_{k+1}) = \text{t.d. } K(a, b)/K(a_1) = \text{t.d. } K(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_h)/K(\bar{a}_1) = 0 \]

with the last 2 equalities following from (i) and the fact that $\deg \bar{R} = 0$, respectively.

Using (2) and (3), one obtains

\[ \text{t.d. } K(a, a_1, \ldots, a_{k+1})/K(a_1, \ldots, a_k) = \text{t.d. } K(a_1, \ldots, a_{k+1}, T^k b)/K(a_1, \ldots, a_k) \geq \text{t.d. } L_k(a_{k+1}, T^k b)/L_k \geq 1. \]

However,

\[ \text{t.d. } K(a, a_1, \ldots, a_{k+1})/K(a, \ldots, a_k) \leq \text{t.d. } K(a, a_1, a_{k+1})/K(a) = \text{t.d. } K(\bar{a}_1, a_1)/K(a) = \deg R = 1. \]

Hence, t.d. $K(a, a_1, \ldots, a_{k+1})/K(a, \ldots, a_k) = 1$ for each $k$. Thus $K(a, a_1, \ldots, a_h)$ is obtained from $R$ by generic prolongations.

Let $\Phi_k: L_k[[t_k]] \to L_k$ be the homomorphism over $L_k$ defined by $\Phi_k t_k = 0$, $0 \leq k \leq h-1$. From the series expansions (1), we note that

\[ a_2^{(i)} = T a_1^{(i)} = T a_1^{(i)} + \sum_{j=1}^{\infty} (T c_{ij}) t_1^j = b_2^{(i)} + \sum_{j=1}^{\infty} (T c_{ij}) t_1^j \]

\[ = \bar{a}_2^{(i)} + \sum_{j=1}^{\infty} d_{2ij} t^j + \sum_{j=1}^{\infty} (T c_{ij}) t_1^j. \]
and thus

$$a_k^{(i)} = \bar{z}_k^{(i)} + \sum_{j=1}^{\infty} d_{kj} t_j^i + \sum_{j=1}^{\infty} (T d_{k-1,j}) t_j^{i+1} + \cdots + \sum_{j=1}^{\infty} (T^{k-1} c_{ij}) t_j^{i+k-2}$$

for $k \geq 2$. $K[a, \cdots, a_{k-1}] \subseteq L_k[[t_k]]$. For $q \leq k$, $a_q \in L_{k-1}[[t_{k-1}]]$ and $\Phi_d a_q = a_q$. But

$$\Phi_k a_k^{(i)} = \Phi_k \left( \bar{z}_k^{(i)} + \sum_{j=1}^{\infty} d_{kj} t_j^i + \cdots + \sum_{j=1}^{\infty} (T^{k-1} d_{ij}) t_j^{i+k-2} \right)$$

$$= \bar{z}_k^{(i)} + \cdots + \sum_{j=1}^{\infty} (T^{k-1} d_{ij}) t_j^{i+k-2} \in L_{k-1}[[t_{k-1}]].$$

Hence $\Phi = \Phi_0 \cdot \Phi_1 \cdots \Phi_{k-1}$ is a well-defined homomorphism on $K[a, \cdots, a_h]$ with $\Phi a_k = \bar{z}_k$, $k = 0, \cdots, h$. Thus

$$(a, \cdots, a_h) \rightarrow (a, \bar{a}_1, \cdots, \bar{a}_h).$$

**Corollary.** Every regular realization of a kernel is the specialization of a principal realization.

**Proof.** By the same argument as in Corollary to Theorem 1 of [2]. (This result has been obtained by other methods [1, p. 189].)

3. **Jacobi's Bound.** Let $K\{y\} = K\{y^{(i)}, \cdots, y^{(n)}\}$ be a polynomial difference ring over a difference field $K$. Let $A_1, \cdots, A_m$ be a system of difference polynomials in $K\{y\}$. Let $r_{ij}$ be the order of $A_i$ in $y^{(j)}$, with $r_{ij} = 0$ if $y^{(j)}$ does not effectively appear in $A_i$. The Jacobi number of the system, $J(A)$, is the maximal diagonal sum of the matrix $|r_{ij}|$ (see [2]).

**Theorem 2.** Let $A_1, \cdots, A_m$ be first order difference polynomials in $K\{y\}$. If $M$ is an irreducible component of $\mathcal{M}(A_1, \cdots, A_m)$ of dimension 0, then $E$ ord $M \leq J(A)$.

**Proof.** We may assume that $K$ is inversive and show that ord $M \leq J(A)$. (See the proof of Theorem IX in [1, Chapter 8].) Let $\bar{a}$ be a generic zero of $\mathcal{M}$. Then $\bar{a}$ is a principal realization of the kernel $\bar{R}$ with field $K(\bar{a}, \bar{a}_1)$ (argument as in Theorem 3 of [2]). $\deg \bar{R} = \dim \mathcal{M} = 0$, and ord $\bar{R} = \text{ord } \mathcal{M} = \text{t.d. } K(\bar{a})/K$.

Since $(\bar{a}, \bar{a}_1)$ is a zero of the ideal $(A_1, \cdots, A_m)$ in $K[y, y_1]$, there is a generic zero of some component of $M(A_1, \cdots, A_m)$ such that

$$(a, a_1) \rightarrow (\bar{a}, \bar{a}_1).$$

Thus t.d. $K(a)/K = t.d. K(\bar{a})/K = s$, and t.d. $K(a_1)/K = t.d. K(\bar{a}_1)/K = r$, for some $s \geq 0$ and $r \geq 0$. 

By two applications of Lemma 4 of [2], one obtains $c$ and $\alpha_1$ such that

$$
\begin{align*}
(a, a_1) &\to (c, \bar{a}_1) &\to (\bar{a}, \alpha_1) &\to (\bar{a}, \bar{a}_1)
\end{align*}
$$

with $t.d. K(a, a_1)/K - t.d. K(c, \bar{a}_1)/K \leq r$ and

$$
t.d. K(c, \bar{a}_1)/K - t.d. K(\bar{a}, \alpha_1)/K \leq s.
$$

Therefore,

$$
t.d. K(a, a_1)/K - t.d. K(\bar{a}, \alpha_1)/K \leq t.d. K(a_1)/K - t.d. K(\bar{a}_1)/K + t.d. K(a)/K - t.d. K(\bar{a})/K.
$$

(4)

Since $(\bar{a}_1)_{K'}(\alpha_1)_{K'}(\bar{a}_1)_{K'}$, $K(\alpha_1) \cong_{K} K(\bar{a}_1)$. Thus, since $K(\bar{a}, \alpha_1)$ is a kernel, $K(\bar{a}, \alpha_1)$ is a kernel $R$. By Theorem 1, there is a principal realization $\alpha$ of $R$ which specializes to $\bar{a}$. But $\alpha$ is a zero of $A_1, \ldots, A_m$ since $(a, a_1)_{K'}(\bar{a}, \alpha_1)$. Hence the specialization $\alpha_{K'} \bar{a}$ is generic, and

$$
K(\bar{a}, \alpha_1) \cong K(\bar{a}, \bar{a}_1).
$$

Using this isomorphism and (4), one obtains

$$
\begin{align*}
\text{ord } M &\leq t.d. K(\bar{a})/K = t.d. K(\bar{a}_1)/K \\
&\leq t.d. K(\bar{a}, \bar{a}_1)/K - t.d. K(\bar{a})/K + t.d. K(a_1)/K \\
&\quad - t.d. K(a, a_1)/K + t.d. K(a)/K \\
&\leq 0 + n - t.d. K(a, a_1)/K(a).
\end{align*}
$$

But, by the same argument as in Theorem 3 of [2], $t.d. K(a, a_1)/K(a) \geq n - J(A)$. Therefore, $\text{ord } M \leq J(A)$.

By an example analogous to that of [2], it can be shown that for any matrix $A$ of nonnegative integers, there exists a system of $n$ difference polynomials in $n$ indeterminates with matrix of orders equal to $A$ such that some zero dimensional component of the variety of the system has effective order equal to $J(A)$. Thus when the Jacobi number is a bound, it is a bound achieved by some system.

**References**


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