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JACOBI'S BOUND FOR FIRST ORDER DIFFERENCE EQUATIONS[1]

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ABSTRACT. Let $A_1, \cdots, A_n$ be a system of difference polynomials in $y^{(1)}, \cdots, y^{(n)}$, and let $\mathscr{M}$ be an irreducible component of the difference variety $\mathscr{M}(A_1, \cdots, A_n)$. If r_{ij} is the order of A_i in $y^{(j)}$, the Jacobi number J of the system is defined to be $\max\{\sum_{i=1}^{n} r_{ij_i}: j_1, \cdots, j_n$ is a permutation of $1, \cdots, n\}$. In this paper it is shown for first order systems that if $\dim \mathscr{M}=0$, then $E \operatorname{ord} \mathscr{M} \leqq J$. The methods used are analogous to those used to obtain the corresponding result for differential equations (given in a recent paper by the author).

1. **Introduction.** For any system of difference polynomials $A_1, \cdots, A_n$, with r_{ij} the order of A_i in $y^{(j)}$, the Ritt number

$$R = \sum_{j=1}^{n} \max\{r_{ij}: i = 1, \cdots, n\}$$

provides a bound for the effective order of an irreducible component $\mathscr{M}$ of dimension 0 [**1**, pp. 253–255]. Greenspan improved this bound for those systems in which every irreducible component has dimension 0. (See [**1**, pp. 256–258].) In the case $n=2$, the Greenspan number coincides with the Jacobi number J. The Jacobi number can also be verified for linear difference systems by a proof analogous to that for the differential case [**3**].

The definitions of [**1**] are assumed, and the notation of [**2**] will be used.

2. **Specialization problem.** Let K be an inversive difference field with automorphism τ. Let $\bar{R}$ be a difference kernel with principal realization $\bar{\alpha}$. In general, if R is a kernel which specializes to $\bar{R}$, there may be no principal realization of R which specializes to $\bar{\alpha}$. (See [**1**, p. 322, Example 2].) The following theorem presents in one case which such a specialization of principal realizations does exist.

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THEOREM 1. *Let K be inversive. Let R and $\bar{R}$ be difference kernels consisting of $K(a, \cdots, a_r)$, T, and $K(\bar{a}, \cdots, \bar{a}_r)$, $\bar{T}$, respectively, $r \geqq 0$, with $K(a, \cdots, a_{r-1}) \cong_K K(\bar{a}, \cdots, \bar{a}_{r-1})$. Let $\bar{\alpha}$ be a principal realization of $\bar{R}$. If R specializes to $\bar{R}$, then there exists a principal realization α of R which specializes to $\bar{\alpha}$.*

PROOF. It may be assumed that the kernels are of length 0 or 1 [**1**, p. 160]. The proof is given for kernels of length 1 (the length 0 case is similar). By an argument like that in Theorem 1 of [**2**], we may assume that $a=\bar{a}$, deg $\bar{R}=0$ and deg $R=1$.

It will be sufficient to show that for any $h>1$ a kernel $K(a, \cdots, a_h)$, T, can be found with $K(a, \cdots, a_{k+1})$ a generic prolongation of $K(a, \cdots, a_k)$, $k=1, \cdots, h-1$, such that $(a, \cdots, a_h)$ specializes to $(\bar{\alpha}, \bar{\alpha}_1, \cdots, \bar{\alpha}_h)$, where $\bar{\alpha}$ is the principal realization of $\bar{R}$. For suppose the theorem is false: then no principal realization of R specializes to $\bar{\alpha}$. Since the number of distinct principal realizations is finite [**1**, p. 156], there exists an integer $H>1$ such that for any principal realization α of R, $(\alpha, \cdots, \alpha_H)$ does not specialize to $(\bar{\alpha}, \cdots, \bar{\alpha}_H)$.

Let L be the algebraic closure of $K\langle\bar{\alpha}\rangle$. By taking free joins and isomorphic kernels if necessary, we may assume that $(\bar{\alpha}, \bar{\alpha}_1)=(a, \bar{a}_1)$, that t.d. $L(a_1)/L=1$, and that $(a, a_1)\overrightarrow{L}(a, \bar{a}_1)$. Then for any given $h>1$, there exist vectors $b_2, \cdots, b_h$ such that

(i) $K(a_1, b_2, \cdots, b_h) \cong_K K(\bar{a}_1, \bar{\alpha}_2, \cdots, \bar{\alpha}_h)$,

(ii) $(a, a_1, b_2, \cdots, b_h)\overrightarrow{L}(a, \bar{a}_1, \bar{\alpha}_2, \cdots, \bar{\alpha}_h)$,

(iii) t.d. $L(a, a_1, b_2, \cdots, b_h)/L=1$

(by Lemma 1 of [**2**] and the isomorphism $K(a_1) \cong_K K(\bar{a}_1)$).

Since (ii) and (iii) hold, Lemma 2 of [**2**] may be applied to obtain parameter $t \in L(a_1, b_2, \cdots, b_h)$, transcendental over L, such that $L[a_1, b_2, \cdots, b_h] \subseteq L[[t]]$ with

$$(1) \qquad \begin{aligned} a_1^{(i)} &= \bar{a}_1^{(i)} + \sum_{j=1}^{\infty} c_{ij} t^j, && i = 1, \cdots, n; \\ b_k^{(i)} &= \bar{\alpha}_k^{(i)} + \sum_{j=1}^{\infty} d_{kij} t^j, && i = 1, \cdots, n;\ k = 2, \cdots, h. \end{aligned}$$

The transformation of the kernel R, $T: K(a) \to K(a_1)$, may be extended to an isomorphism T of $K(a, \bar{a}_1, \bar{\alpha}_2, \cdots, \bar{\alpha}_{h-1})$ onto $K(a_1, b_2, \cdots, b_h)$, obtained by the composition of $\bar{T}$ and the isomorphism of (i):

$$T: K(a, \bar{a}_1, \cdots, \alpha_{h-1}) \xrightarrow{\bar{T}} K(\bar{a}_1, \bar{\alpha}_2, \cdots, \bar{\alpha}_h) \xrightarrow[K]{} K(a_1, b_2, \cdots, b_h).$$

T may then be extended to a monomorphism of L into a field L_1, which is the algebraic closure of the quotient field of $L[[t]]$. Then T extends to a monomorphism of $L[[t]]$ into $L_1[[t_1]]$, where t_1 is transcendental over L_1 and $T(t)=t_1$. Repeating this method of extension, for each $k=2, \cdots, h-1$, one may extend T to a monomorphism $T: L_{k-1}[[t_{k-1}]] \to L_k[[t_k]]$, with $L_{k-1}[[t_{k-1}]] \subseteq L_k$ and t_k transcendental over L_k.

Since $a_1 \in L[[t]]$, $T^k a = T^{k-1} a_1 \in L_{k-1}[[t_{k-1}]]$, $k=2, \cdots, h$. Let $a_k = T^k a$. Then $K[a, a_1, \cdots, a_{h-1}] \subseteq L_{h-2}[[t_{h-2}]]$, and T restricted to $K[a, a_1, \cdots, a_h]$ is an isomorphism onto $K[a_1, \cdots, a_h]$. Thus $K(a, a_1, \cdots, a_h)$ is a kernel which can be obtained from R by $h-1$ prolongations.

In the following let b denote the vector $(b_2, \cdots, b_h)$. It will be shown that $K(a, a_1, \cdots, a_h)$ is obtained by generic prolongations. Since $t \in L(a_1, b)$, $t_k = T^k t \in L_k(a_{k+1}, T^k b)$, $1 \leqq k \leqq h-1$. But t_k is transcendental over L_k; thus

$$\text{t.d. } L_k(a_{k+1}, T^k b)/L_k \geqq 1. \tag{2}$$

It may be noted that

$$\begin{aligned} &\text{t.d. } K(a, \cdots, a_{k+1}, T^k b)/K(a, \cdots, a_{k+1}) \\ &\qquad \leqq \text{t.d. } K(a_{k+1}, T^k b)/K(a_{k+1}) = \text{t.d. } K(a_1, b)/K(a_1) \\ &\qquad = \text{t.d. } K(\bar{a}_1, \bar{\alpha}_2, \cdots, \bar{\alpha}_h)/K(\bar{a}_1) = 0 \end{aligned} \tag{3}$$

with the last 2 equalities following from (i) and the fact that deg $\bar{R}=0$, respectively.

Using (2) and (3), one obtains

$$\begin{aligned} &\text{t.d. } K(a, \cdots, a_{k+1})/K(a, \cdots, a_k) \\ &\qquad = \text{t.d. } K(a, \cdots, a_{k+1}, T^k b)/K(a, \cdots, a_k) \\ &\qquad \geqq \text{t.d. } L_k(a_{k+1}, T^k b)/L_k \geqq 1. \end{aligned}$$

However,

$$\begin{aligned} \text{t.d. } K(a, \cdots, a_{k+1})/K(a, \cdots, a_k) &\leqq \text{t.d. } K(a_k, a_{k+1})/K(a_k) \\ &= \text{t.d. } K(a, a_1)/K(a) = \deg R = 1. \end{aligned}$$

Hence, t.d. $K(a, \cdots, a_{k+1})/K(a, \cdots, a_k) = 1$ for each k. Thus $K(a, \cdots, a_h)$ is obtained from R by generic prolongations.

Let $\Phi_k: L_k[[t_k]] \to L_k$ be the homomorphism over L_k defined by $\Phi_k t_k = 0$, $0 \leqq k \leqq h-1$. From the series expansions (1), we note that

$$\begin{aligned} a_2^{(i)} = T a_1^{(i)} = T \bar{a}_1^{(i)} + \sum_{j=1}^{\infty} (T c_{ij}) t_1^j &= b_2^{(i)} + \sum_{j=1}^{\infty} (T c_{ij}) t_1^j \\ = \bar{\alpha}_2^{(i)} + \sum_{j=1}^{\infty} d_{2ij} t^j + \sum_{j=1}^{\infty} (T c_{ij}) t_1^j. \end{aligned}$$

and thus

$$a_k^{(i)} = \bar{\alpha}_k^{(i)} + \sum_{j=1}^{\infty} d_{kij} t^j + \sum_{j=1}^{\infty} (T d_{k-1ij}) t_1^j + \cdots + \sum_{j=1}^{\infty} (T^{k-1} c_{ij}) t_{k-1}^j,$$

for $k \geqq 2$. $K[a, \cdots, a_{k+1}] \subseteq L_k[[t_k]]$. For $q \leqq k$, $a_q \in L_{k-1}[[t_{k-1}]]$ and $\Phi_k a_q = a_q$. But

$$\Phi_k a_{k+1}^{(i)} = \Phi_k \left(\bar{\alpha}_{k+1}^{(i)} + \sum_{j=1}^{\infty} d_{k+1ij} t^j + \cdots + \sum_{j=1}^{\infty} (T^k c_{ij}) t_k^j \right)$$

$$= \bar{\alpha}_{k+1}^{(i)} + \cdots + \sum_{j=1}^{\infty} (T^{k-1} d_{2ij}) t_{k-1}^j \in L_{k-1}[[t_{k-1}]].$$

Hence $\Phi = \Phi_0 \cdot \Phi_1 \cdot \cdots \cdot \Phi_{h-1}$ is a well-defined homomorphism on $K[a, \cdots, a_h]$ with $\Phi a_k = \bar{\alpha}_k$, $k = 0, \cdots, h$. Thus

$$(a, \cdots, a_h) \xrightarrow[K]{} (a, \bar{a}_1, \cdots, \bar{\alpha}_h).$$

COROLLARY. *Every regular realization of a kernel is the specialization of a principal realization.*

PROOF. By the same argument as in Corollary to Theorem 1 of [**2**]. (This result has been obtained by other methods [**1**, p. 189].)

3. **Jacobi's bound.** Let $K\{y\} = K\{y^{(1)}, \cdots, y^{(n)}\}$ be a polynomial difference ring over a difference field K. Let $A_1, \cdots, A_m$ be a system of difference polynomials in $K\{y\}$. Let r_{ij} be the order of A_i in $y^{(j)}$, with $r_{ij} = 0$ if $y^{(j)}$ does not effectively appear in A_i. The Jacobi number of the system, $J(A)$, is the maximal diagonal sum of the matrix $|r_{ij}|$ (see [**2**]).

THEOREM 2. *Let $A_1, \cdots, A_m$ be first order difference polynomials in $K\{y\}$. If $\mathcal{M}$ is an irreducible component of $\mathcal{M}(A_1, \cdots, A_m)$ of dimension* 0, *then E* ord $\mathcal{M} \leqq J(A)$.

PROOF. We may assume that K is inversive and show that ord $\mathcal{M} \leqq J(A)$. (See the proof of Theorem IX in [**1**, Chapter 8].) Let $\bar{\alpha}$ be a generic zero of $\mathcal{M}$. Then $\bar{\alpha}$ is a principal realization of the kernel $\bar{R}$ with field $K(\bar{\alpha}, \bar{\alpha}_1)$ (argument as in Theorem 3 of [**2**]). deg $\bar{R} = \dim \mathcal{M} = 0$, and ord $\bar{R} =$ ord $\mathcal{M} =$ t.d. $K(\bar{\alpha})/K$.

Since $(\bar{\alpha}, \bar{\alpha}_1)$ is a zero of the ideal $(A_1, \cdots, A_m)$ in $K[y, y_1]$, there is a generic zero of some component of $M(A_1, \cdots, A_m)$ such that

$$(a, a_1) \xrightarrow[K]{} (\bar{\alpha}, \bar{\alpha}_1).$$

Thus t.d. $K(a)/K -$ t.d. $K(\bar{\alpha})/K = s$, and t.d. $K(a_1)/K -$ t.d. $K(\bar{\alpha}_1)/K = r$, for some $s \geqq 0$ and $r \geqq 0$.

By two applications of Lemma 4 of [**2**], one obtains c and α_1 such that

$$(a, a_1) \underset{K}{\longrightarrow} (c, \bar{\alpha}_1) \underset{K}{\longrightarrow} (\bar{\alpha}, \alpha_1) \underset{K}{\longrightarrow} (\bar{\alpha}, \bar{\alpha}_1)$$

with t.d. $K(a, a_1)/K -$ t.d. $K(c, \bar{\alpha}_1)/K \leqq r$ and

$$\text{t.d. } K(c, \bar{\alpha}_1)/K - \text{t.d. } K(\bar{\alpha}, \alpha_1)/K \leqq s.$$

Therefore,

$$\begin{aligned} &\text{t.d. } K(a, a_1)/K - \text{t.d. } K(\bar{\alpha}, \alpha_1)/K \\ (4) \qquad &\leqq \text{t.d. } K(a_1)/K - \text{t.d. } K(\bar{\alpha}_1)/K + \text{t.d. } K(a)/K - \text{t.d. } K(\bar{\alpha})/K. \end{aligned}$$

Since $(\bar{\alpha}_1)\overrightarrow{_K}(\alpha_1)\overrightarrow{_K}(\bar{\alpha}_1)$, $K(\alpha_1) \cong_K K(\bar{\alpha}_1)$. Thus, since $K(\bar{\alpha}, \bar{\alpha}_1)$ is a kernel, $K(\bar{\alpha}, \alpha_1)$ is a kernel R. By Theorem 1, there is a principal realization α of R which specializes to $\bar{\alpha}$. But α is a zero of $A_1, \cdots, A_m$ since $(a, a_1)\overrightarrow{_K}(\bar{\alpha}, \alpha_1)$. Hence the specialization $\alpha\overrightarrow{_K}\bar{\alpha}$ is generic, and

$$K(\bar{\alpha}, \alpha_1) \cong K(\bar{\alpha}, \bar{\alpha}_1).$$

Using this isomorphism and (4), one obtains

$$\begin{aligned} \text{ord } \mathcal{M} &= \text{t.d. } K(\bar{\alpha})/K = \text{t.d. } K(\bar{\alpha}_1)/K \\ &\leqq \text{t.d. } K(\bar{\alpha}, \bar{\alpha}_1)/K - \text{t.d. } K(\bar{\alpha})/K + \text{t.d. } K(a_1)/K \\ &\quad - \text{t.d. } K(a, a_1)/K + \text{t.d. } K(a)/K \\ &\leqq 0 + n - \text{t.d. } K(a, a_1)/K(a). \end{aligned}$$

But, by the same argument as in Theorem 3 of [**2**], t.d. $K(a, a_1)/K(a) \geqq n - J(A)$. Therefore, ord $\mathcal{M} \leqq J(A)$.

By an example analogous to that of [**2**], it can be shown that for any matrix A of nonnegative integers, there exists a system of n difference polynomials in n indeterminates with matrix of orders equal to A such that some zero dimensional component of the variety of the system has effective order equal to $J(A)$. Thus when the Jacobi number is a bound, it is a bound achieved by some system.

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