RINGS SATISFYING MONOMIAL IDENTITIES

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ABSTRACT. The following theorem is proved: Suppose $R$ is an associative ring and suppose that $w(x_1, \ldots, x_n)$ is a fixed word distinct from $x_1 \cdots x_n$. If, further, $x_1 \cdots x_n = w(x_1, \ldots, x_n)$, for all $x_1, \ldots, x_n$ in $R$, then the commutator ideal of $R$ is nilpotent. Moreover, it is shown that this theorem need not be true if the word $w$ is not fixed.

Suppose $R$ is an associative ring and suppose $x_1, \ldots, x_n$ are elements of $R$. A word $w(x_1, \ldots, x_n)$ in $x_1, \ldots, x_n$ is a product in which each factor is $x_i$ for some $i = 1, \ldots, n$. Our present object is to prove

**Theorem 1.** Suppose $R$ is an associative ring and suppose $w(x_1, \ldots, x_n)$ is a fixed word distinct from the word $x_1 \cdots x_n$. Suppose

1. $x_1 \cdots x_n = w(x_1, \ldots, x_n)$, for all $x_1, \ldots, x_n$ in $R$.

Then there exists a positive integer $m$ such that $R^mC(R)R^m = (0)$, where $C(R)$ is the commutator ideal of $R$. In particular, the commutator ideal of $R$ is nilpotent.

Moreover, a counterexample is given which shows that Theorem 1 need not be true if $w(x_1, \ldots, x_n)$ is not a fixed word.

In preparation for the proof of Theorem 1, we first show the following lemmas.

**Lemma 1.** Suppose $R$ is an associative semisimple ring, and suppose $w(x_1, \ldots, x_n)$ is a fixed word involving each of the elements $x_1, \ldots, x_n$ of $R$. If, further,

2. $x_1 \cdots x_n = w(x_1, \ldots, x_n)$, for all $x_1, \ldots, x_n$ in $R$,

then $R$ is commutative.

**Proof.** Suppose, first, that $R$ has an identity $1$. We now distinguish two cases.

**Case 1.**

3. $x_1 \cdots x_n = w(x_1, \ldots, x_n) = x_{\sigma(1)} \cdots x_{\sigma(n)}$.
where \( \sigma \) is a permutation of \( \{1, \cdots, n\} \) distinct from the identity permutation. Then, for some integers \( i, j \), we have \( i < j \) but \( \sigma(i) > \sigma(j) \). Now, let \( a, b \in R \), and set in (3), \( x_{\sigma(i)} = a, x_{\sigma(j)} = b, \) \( x_k = 1 \) for all \( k \neq \sigma(i), k \neq \sigma(j) \), we get \( ba = ab \), and the lemma follows.

Case 2.

(4) \[ x_1 \cdots x_n = w(x_1, \cdots, x_n), \]

some \( x_i \) appears at least twice in \( w(x_1, \cdots, x_n) \).

In this case, by setting \( x_1 = \cdots = x_{t-1} = x_{t+1} = \cdots = x_n = 1 \) in (4), we get

(5) \[ x_i = x_i^k, \quad \text{for all } x_i \in R \quad (k > 1). \]

Hence \cite[p. 217]{2}, \( R \) is commutative, and the lemma follows again.

Returning to the general case, observe that, since \( R \) is semisimple, \( R \) is isomorphic to a subdirect sum of primitive rings \( R_i, i \in \Gamma \), each of which clearly satisfies (2). Since every subring and every homomorphic image of \( R \) satisfies (2), it follows \cite[p. 33]{2} that some complete matrix ring, \( \Delta_m \), over a division ring satisfies (2) also. Since \( \Delta_m \) has an identity, it follows (by the first part of this proof) that \( \Delta_m \) is commutative. Thus \( m = 1 \), and \( \Delta_m = \Delta \) is a field. Hence \cite[p. 33]{2} the primitive ring \( R \) is isomorphic to the field \( \Delta \). Thus \( R \) is isomorphic to a subdirect sum of fields, and hence \( R \) is commutative. This proves the lemma.

Next, we consider the case in which the word \( w(x_1, \cdots, x_n) \) satisfies (4). In this case, we can even say more. Indeed, we have

**Lemma 2.** Suppose \( R \) is an associative ring and suppose that \( C(R) \) and \( J \) denote the commutator ideal and Jacobson ideal of \( R \). Suppose that \( w(x_1, \cdots, x_n) \) is a fixed word involving each of the elements \( x_1, \cdots, x_n \) of \( R \). Suppose, moreover, that for some \( t, x_t \) appears at least twice in \( w(x_1, \cdots, x_n) \). If, further,

(6) \[ x_1 \cdots x_n = w(x_1, \cdots, x_n), \quad \text{for all } x_1, \cdots, x_n \in R, \]

then (i) \( R/J \) is isomorphic to a subdirect sum of finite fields of orders bounded by the length of \( w \); (ii) \( C(R) \subseteq J \); (iii) \( J \) consists of precisely the set of nilpotent elements of \( R \).

**Proof.** Since \( R/J \) is a semisimple ring which, clearly, satisfies (6), it follows, by Lemma 1, that \( R/J \) is commutative, and hence \( R/J \) is isomorphic to a subdirect sum of fields \( F_i, i \in \Gamma \). Now, each \( F_i \) clearly satisfies (6), and hence by setting \( x_i = 1 \) for all \( i \neq t \) in (6), we obtain

(7) \[ x_i = x_i^k, \quad \text{for all } x_i \in R \quad (k > 1). \]

Therefore \( F_t \) is a finite field with at most \( k \) elements, and, clearly, \( k \) is equal to or less than the length of the word \( w(x_1, \cdots, x_n) \). This proves (i).
Part (ii) follows at once, since $R/J$ is commutative. Finally, to prove (iii), suppose $a \in J$, and set $x_i = a$, for all $i$, in (6). We get, $a^n = a^n a^l$ for some $l \geq 1$, and hence $a^n = 0$. Conversely, if $a$ is nilpotent, then $\bar{a} = (a + J)$ is a nilpotent element in $R/J$, and hence by (i), $\bar{a} = 0$. Thus $a \in J$, and the lemma is proved.

Next, we prove

**Lemma 3.** Suppose $R$ is an associative ring, and suppose $J$ is the Jacobson radical of $R$. Suppose that $w(x_1, \cdots, x_n)$ is a fixed word involving each of the elements $x_1, \cdots, x_n$ of $R$ and in which some $x_i$ appears at least twice. Suppose, moreover, that

\[ x_1 \cdots x_n = w(x_1, \cdots, x_n), \quad \text{for all } x_1, \cdots, x_n \text{ in } R. \tag{8} \]

Then, for some $i$, $1 \leq i \leq n$, we have $R^{i-1}JR^{n-i} = (0)$.

**Proof.** By Lemma 2 (iii), $J$ is a nil ring. Now, let $a \in J$, and set in (8), $x_1 = \cdots = x_n = a$, we get $a^n = a^n a^l$, for some $l \geq 1$. Therefore the nil ring $J$ satisfies $a^k = 0$, and thus [1, p. 28] $J$ is locally nilpotent. Next, let $a_1, \cdots, a_n \in J$. Then the ring generated by $a_1, \cdots, a_n$ is nilpotent, say of index $k$. Now, by reiterating (8) until the length of the word $w(x_1, \cdots, x_n)$ in the right-hand side becomes $\geq k$, it follows that $a_1 \cdots a_n = 0$, and hence $J^n = (0)$. Next, since $x_i$ appears at least twice in the word $w(x_1, \cdots, x_n)$, we can, by reiterating in (8), obtain a word $w'(x_1, \cdots, x_n)$ of length $\geq n^2$ and such that

\[ x_1 \cdots x_n = w'(x_1, \cdots, x_n), \quad \text{for all } x_1, \cdots, x_n \text{ in } R. \tag{9} \]

Observe that in the word $w'(x_1, \cdots, x_n)$, some $x_i$ appears at least $n$ times. We now fix $i$, and substitute $x_i = a$; $x_j = r_j$, $j \neq i$, where each $r_j \in R$, we get

\[ r_1 \cdots r_{i-1} a r_{i+1} \cdots r_n \in J^n = (0). \]

Hence, $R^{i-1}JR^{n-i} = (0)$, and the lemma is proved.

Our final lemma is true for semigroups (and hence, *a fortiori*, for rings), and has been proved in [4, Theorem 1].

**Lemma 4.** Let $S$ be a semigroup such that, for all $x_1, \cdots, x_n$ in $S$,

\[ x_1 \cdots x_n = x_{\sigma(1)} \cdots x_{\sigma(n)}, \]

where $\sigma$ is a fixed permutation of $\{1, \cdots, n\}$ distinct from the identity permutation. Then there exists an integer $m$ such that, for all $u, v$ in $S^m$ and all $x, y$ in $S$, we have $uxyv = uyxv$.

We are now in a position to prove Theorem 1.
Proof of Theorem 1. First, suppose the word $w(x_1, \cdots, x_n)$ does not involve $x_i$, for some $i$. In (1), set $x_i=0$ and, for $j \neq i$, let $x_j$ be arbitrary; we get $w(x_1, \cdots, x_n) \equiv 0$ and hence, by (1), $x_1 \cdots x_n = 0$ for all $x_1, \cdots, x_n$ in $R$ (since $w(x_1, \cdots, x_n)$ is fixed). Thus $R^n = (0)$, and Theorem 1 follows at once. Next, suppose $w(x_1, \cdots, x_n) = x_{\sigma(1)} \cdots x_{\sigma(n)}$, for some permutation $\sigma$ of $\{1, \cdots, n\}$ different from the identity. Then, by Lemma 4,

$$u(xy - yx)v = 0, \quad \text{for all } u, v \in R^n \text{ and all } x, y \in R.$$  

Hence, $R^n C(R) R^n = (0)$, and Theorem 1 follows again. The only case left is when $w(x_1, \cdots, x_n)$ involves each $x_i$ and, moreover, some $x_i$ appears at least twice in $w(x_1, \cdots, x_n)$. By Lemmas 2 and 3 we have $R^{i-1} C(R) R^{n-i} \subseteq R^{i-1} J R^{n-i} = (0)$, for some $i$, $1 \leq i \leq n$, and once again the theorem follows. This completes the proof.

Corollary. Suppose $R$ is an associative semiprime ring satisfying the hypothesis of Theorem 1. Then $R$ is commutative.

Proof. Since $R$ is a semiprime ring, the prime radical of $R$ is $(0)$ [3, p. 146], and hence $R$ contains no nonzero nilpotent ideals. Now, by Theorem 1, the commutator ideal, $C(R)$, of $R$ is nilpotent, and hence $C(R) = (0)$. Therefore $R$ is commutative, and the corollary is proved.

We conclude with the following

Remark. Theorem 1 need not be true if we replace the fixed word $w(x_1, \cdots, x_n)$ by a “variable” word (depending on $x_1, \cdots, x_n$). For, suppose $R$ is the complete matrix ring, $(GF(2))_2$, of all $2 \times 2$ matrices over $GF(2)$. It is easily verified that

$$x_1 x_2 = x_1 x_2^2 \quad \text{if } x_1 \text{ is invertible or idempotent},$$

$$= x_1 x_2^7 \quad \text{if } x_2 \text{ is invertible or idempotent},$$

$$= (x_1 x_2)^2 \quad \text{otherwise.}$$

However, the commutator ideal of $(GF(2))_2$ is not even nil. In verifying (10), observe that (i) $x^8 = x^8$ holds in $(GF(2))_2$; (ii) every matrix in $(GF(2))_2$ is invertible, or idempotent, or nilpotent; (iii) the product of any two nilpotent matrices in $(GF(2))_2$ is idempotent.

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References


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