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A CONDITION FOR A FINITE GROUP TO BE CYCLIC

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ABSTRACT. A finite group, with the property that for every prime power $q=p^k$ there are at most $p^{k+1}-1$ elements in the group whose qth power is the identity, is a cyclic group.

It seems to be well known that a finite group, which need not be given to be abelian, must be cyclic if for every positive integer n, the number of elements for which x^n is the identity does not exceed n. The object of this note is to prove the stronger

THEOREM. *Let G be a finite group, with identity e, such that for every prime power $q=p^k$ there exist at most $p^{k+1}-1$ elements satisfying $x^q=e$. Then G is cyclic.*

PROOF. Suppose that $p|o(G)$, with say $p^m||o(G)$. Let H denote any Sylow p-subgroup. Then choosing $q=p^{m-1}$ it follows that H is cyclic, since it has at least one element whose order does not divide q.

Furthermore, H is the only Sylow p-subgroup, and so is a normal subgroup of G. For in the contrary case, there would exist at least $p+1$ distinct cyclic Sylow p-subgroups, no two of which could contain an element of order p^m in common. Thus they would contain together at least $(p+1)p^{m-1}(p-1)+p^{m-1}=p^{m+1}$ distinct elements, all of which would satisfy $x^q=e$ with $q=p^m$, contrary to hypothesis.

Thus G must be the direct product of its Sylow subgroups, each one of which is cyclic, and accordingly G must be cyclic.

COROLLARY. *Every noncyclic finite group has for at least one prime power p^k, at least $p^{k-1}(p^2-1)$ elements of order p^k exactly.*

PROOF. Suppose on the contrary that for every prime power p^k there were less than $p^{k-1}(p^2-1)$ elements of order p^k. Now the number of such elements is divisible by $p^{k-1}(p-1)$, for each one generates a cyclic subgroup with this number of generators. Thus there can be at most $p^k(p-1)$ elements of order p^k for every prime power. But then there would be at most $p^{k+1}-p+1\leqq p^{k+1}-1$ elements satisfying $x^q=e$ for every prime power $q=p^k$, and the group would be cyclic by the theorem.

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