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DERIVATION OF THE HOPF-COLE SOLUTION TO BURGERS' EQUATION BY STOCHASTIC INTEGRALS[1]

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Abstract. In this paper the Hopf-Cole solution to Burgers' equation is derived by use of stochastic integrals. First the equation is written in Hamilton-Jacobi form, and then, following an idea of Freidlin, the solution is differentiated along a Brownian motion.

Let u be a solution to the (backwards) Cauchy problem for Burgers' equation

$$u_t + uu_x + \tfrac{1}{2} u_{xx} = 0, \qquad t < T,\ x \in \boldsymbol{R},$$
$$u(T, x) = g(x),$$

where g is bounded and measurable. In conservation form, the equation becomes

$$u_t + \tfrac{1}{2}(u^2 + u_x)_x = 0.$$

We make the change of variables $u=\phi_x$; the motivation for considering ϕ comes from Hamilton-Jacobi theory. Then the equation can be integrated with respect to x to yield

$$\phi_t + \tfrac{1}{2}\phi_x^2 + \tfrac{1}{2}\phi_{xx} = 0, \qquad t < T,\ x \in \boldsymbol{R},$$
$$\phi(T, x) = G(x),$$

where $G' = g$. (Here we have if necessary added to ϕ an appropriate function of time so that the equation is rendered homogeneous.) We now consider the stochastic differential of ϕ along the diffusion governed by the linear part of the operator, i.e., the Brownian motion

$$y_r = z + b_r - b_s, \qquad s \leqq r \leqq T,$$

where b is a standard Brownian motion starting at zero and $z \in \boldsymbol{R}$ and $s < T$ are fixed. Then by Ito's lemma [**1**, p. 32]

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\begin{aligned}
d\phi(r, y_r) &= \phi_r\,dr + \phi_x\,dy_r + \tfrac{1}{2}\phi_{xx}(dy_r)^2 \\
&= \phi_r\,dr + \phi_x\,db_r + \tfrac{1}{2}\phi_{xx}\,dr \\
&= \phi_x\,db_r - \tfrac{1}{2}\phi_x^2\,dr, \qquad s < r < T.
\end{aligned} \tag{1}
$$

We recognize the right-hand side as the differential occuring in the integrand of the exponential martingale [**1**, p. 25]

$$Z(t) = \exp\left\{\int_s^t \phi_x\,db_r - \frac{1}{2}\int_s^t \phi_x^2\,dr\right\}.$$

(In general Z would be only a supermartingale, but from the maximum principle we know ϕ_x is bounded.) This suggests integrating (1) over (s, T):

$$\phi(T, y_T) - \phi(s, y_s) = \log Z(T),$$

and then taking the exponential of both sides,

$$\exp[\phi(T, y_T) - \phi(s, y_s)] = Z(T).$$

Note that $y_s \equiv z$ and $\phi(T, x) = G(x)$, hence this becomes

$$\exp[G(y_T) - \phi(s, z)] = Z(T).$$

But since Z is a martingale the expectation of $Z(t)$ is constant and equal to unity, hence

$$\exp[\phi(s, z)] = E\exp[G(y_T)].$$

Next take the logarithm of both sides and differentiate with respect to z:

$$u(s, z) = \frac{\partial}{\partial z}\log E\exp[G(y_T)] = \frac{\partial}{\partial z}\log H_{T-s}\exp[G(z)]$$

where H_t stands for convolution with the heat kernel at time t. This is the Hopf-Cole solution. (See [**2**].)

For completeness we should mention another use of the martingale Z. It is easy to show, by considering the differential of $t \to u(t, y_t)Z(t)$ that

$$u(s, z) = E[g(y_T)Z(T)], \qquad s < T. \tag{2}$$

(This is actually a form of the generalized Cameron-Martin formula. See [**1**, p. 67].) Representations such as this were used by Freidlin [**3**] to prove existence of a global solution to a wide class of quasilinear Cauchy problems.

In a direction converse to that in this note, Henry McKean has started with the Hopf-Cole solution and given it a probabilistic interpretation. This work is unpublished.

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