DERIVATION OF THE HOPF-COLE SOLUTION TO BURGERS' EQUATION BY STOCHASTIC INTEGRALS

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Abstract. In this paper the Hopf-Cole solution to Burgers’ equation is derived by use of stochastic integrals. First the equation is written in Hamilton-Jacobi form, and then, following an idea of Freidlin, the solution is differentiated along a Brownian motion.

Let \( u \) be a solution to the (backwards) Cauchy problem for Burgers’ equation

\[
    u_t + uu_x + \frac{1}{2} u_{xx} = 0, \quad t < T, \quad x \in \mathbb{R},
\]

\[
    u(T, x) = g(x),
\]

where \( g \) is bounded and measurable. In conservation form, the equation becomes

\[
    u_t + \frac{1}{2}(u^2 + u_x)_x = 0.
\]

We make the change of variables \( u = \phi_x \); the motivation for considering \( \phi \) comes from Hamilton-Jacobi theory. Then the equation can be integrated with respect to \( x \) to yield

\[
    \phi_t + \frac{1}{2} \phi_x^2 + \frac{1}{2} \phi_{xx} = 0, \quad t < T, \quad x \in \mathbb{R},
\]

\[
    \phi(T, x) = G(x),
\]

where \( G' = g \). (Here we have if necessary added to \( \phi \) an appropriate function of time so that the equation is rendered homogeneous.) We now consider the stochastic differential of \( \phi \) along the diffusion governed by the linear part of the operator, i.e., the Brownian motion

\[
    y_r = z + b_r - b_s, \quad s \leq r \leq T,
\]

where \( b \) is a standard Brownian motion starting at zero and \( z \in \mathbb{R} \) and \( s < T \) are fixed. Then by Ito’s lemma [1, p. 32]
\[ d\phi(r, y_r) = \phi_r dr + \phi_x dy_x + \frac{1}{2} \phi_{xx}(dy_x)^2 \]

\[ = \phi_r dr + \phi_x db_r + \frac{1}{2} \phi_{xx} dr \]

\[ = \phi_x db_r - \frac{1}{2} \phi_x^2 dr, \quad s < r < T. \]

We recognize the right-hand side as the differential occurring in the integrand of the exponential martingale [1, p. 25]

\[ Z(t) = \exp \left( \int_t^s \phi_x db_r - \frac{1}{2} \int_t^s \phi_x^2 dr \right). \]

(In general Z would be only a supermartingale, but from the maximum principle we know \( \phi_x \) is bounded.) This suggests integrating (1) over \((s, T)\):

\[ \phi(T, y_T) - \phi(s, y_s) = \log Z(T), \]

and then taking the exponential of both sides,

\[ \exp[\phi(T, y_T) - \phi(s, y_s)] = Z(T). \]

Note that \( y_s \equiv z \) and \( \phi(T, x) = G(x) \), hence this becomes

\[ \exp[G(y_T) - \phi(s, z)] = Z(T). \]

But since \( Z \) is a martingale the expectation of \( Z(t) \) is constant and equal to unity, hence

\[ \exp[\phi(s, z)] = E \exp[G(y_T)]. \]

Next take the logarithm of both sides and differentiate with respect to \( z \):

\[ u(s, z) = \frac{\partial}{\partial z} \log E \exp[G(y_T)] = \frac{\partial}{\partial z} \log H_{T-s} \exp[G(z)] \]

where \( H_t \) stands for convolution with the heat kernel at time \( t \). This is the Hopf-Cole solution. (See [2].)

For completeness we should mention another use of the martingale \( Z \). It is easy to show, by considering the differential of \( t \rightarrow u(t, y_t)Z(t) \) that

\[ u(s, z) = E[g(y_T)Z(T)], \quad s < T. \]

(This is actually a form of the generalized Cameron-Martin formula. See [1, p. 67].) Representations such as this were used by Freidlin [3] to prove existence of a global solution to a wide class of quasilinear Cauchy problems.

In a direction converse to that in this note, Henry McKean has started with the Hopf-Cole solution and given it a probabilistic interpretation. This work is unpublished.
REFERENCES


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