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PARTIALLY-ORTHOGONAL POLYNOMIALS[1]

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ABSTRACT. This paper contains a discussion of partially-orthogonal polynomials. This is an extension of the concept of quasi-orthogonal polynomials. Some relationships between various partially-orthogonal polynomials are obtained. The concept of pseudo-polynomials is defined and used as an example of partially-orthogonal polynomials. Polynomials obtained from the simple Laguerre polynomials are also used as an example.

The concept of quasi-orthogonal polynomials is discussed by Dickinson **[2]** and by Chihara **[1]**. It is the purpose of this paper to discuss some generalizations of the concept of quasi-orthogonal polynomials and to obtain recurrence relations between the various polynomials. Some examples will be given. Also, the concept of polynomials will be generalized.

DEFINITION 1. Let $\{Q_n(x, m)\}_{n=0}^{\infty}$ be a set of polynomials, where each $Q_n(x, m)$ is of degree n. The $Q_n(x, m)$ will be called partially-orthogonal of deficiency m if there exists an interval (a, b) such that

$$\begin{aligned}\int_a^b w(x)x^jQ_k(x, m)\,dx &= 0 \quad \text{for } 0 < j < k - m,\ k > m,\\ &\neq 0 \quad \text{for } j > k - m,\ k > m,\end{aligned}$$

where $w(x)$ is a nonnegative weight function. If $m=0$ the set of polynomials are fully orthogonal. If $m=1$ the set of polynomials are quasi-orthogonal. The $m=0$ index will be omitted in this paper. For simplicity all the examples of polynomials here will have leading coefficient unity, and this is assumed throughout the paper.

DEFINITION 2. We will call two partially-orthogonal sets of polynomials related if the weight function and interval of integration are the same but the deficiencies are different.

DEFINITION 3. Two polynomials will be said to share the same zero if they are both *annihilated* by the same *operation*. By operation is meant any linear functional F, and F annihilates the polynomial Q if $F(Q)=0$.

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THEOREM 1. *For two sets of related polynomials there exist nonzero constants* $\{A_{n,m}\}_{n=m+1}^{\infty}$ *such that* $Q_n(x, m)=Q_n(x, k)+A_{n,m}Q_{n-1}(x, k)$, *where* $k=m-1$ *and* $n>m$, *providing* $Q_n(x, m)$ *and* $Q_j(x, k)$ *share* j *zeros for* $j<n$ *and share* $n-1$ *for* $j=n$. *However, only* k *zeros will be due to nonorthogonality conditions.*

PROOF. The following expansion is valid (see Gould [3], Rowe [4] or Tauber [5]).

$$Q_n(x, m) = \sum_{j=0}^{n} A_n^j Q_j(x, k), \qquad k = m - 1.$$

Under the stated conditions all but A_n^{n-1} and A_n^n are zero. Therefore, $A_n^n=1$ and $A_n^{n-1}=A_{n,m}$.

COROLLARY 1. *Let* $\{Q_n(x, L)\}_{n=0}^{\infty}$ *and* $\{Q_n(x, m)\}_{n=0}^{\infty}$ *be two sets of partially orthogonal polynomials where* $m-L=c$ *(a positive integer). If for* L *and* m *and all intermediate deficiencies, polynomials whose deficiencies differ by one satisfy Theorem* 1, *then, for* $n>m$,

$$Q_n(x, m) = Q_n(x, L) + \sum_{j=1}^{c} c_{n,j} Q_{n-j}(x, L).$$

PROOF. This result can be obtained by substitution and the relations between the $c_{n,j}$'s and the $A_{n,m}$'s can be obtained by induction.

THEOREM 2. *For any set of partially-orthogonal polynomials of deficiency* m *there is more than one related set of partially-orthogonal polynomials of deficiency* $m+1$.

PROOF. Since the nth degree polynomial of deficiency m will satisfy one more orthogonality condition than the nth degree polynomial of deficiency $m+1$, the nth degree polynomial of deficiency $m+1$ will have a nonshared zero and this nonshared zero can be varied resulting in more than one set of polynomials of deficiency $m+1$. In the above discussion n must be greater than the deficiency for the polynomial to have an orthogonality condition.

COROLLARY 2. *Given*

$$\{Q_n(x, m)\}_{n=0}^{\infty}, \qquad \{Q_n(x, m+1)\}_{n=0}^{\infty} \quad \textit{and} \quad \{\bar{Q}_n(x, m+1)\}_{n=0}^{\infty}$$

where the first and second sets and the first and third sets satisfy Theorem 1, *then* $Q_n(x, m+1)=\bar{Q}_n(x, m+1)+B_{n,m}Q_{n-1}(x, m)$.

PROOF. Since $Q_n(x, m+1)=Q_n(x, m)+A_{n,m+1}Q_{n-1}(x, m)$ and

$$\bar{Q}_n(x, m+1)=Q_n(x, m)+\bar{A}_{n,m+1}Q_{n-1}(x, m)$$

the result follows easily. Many other easily obtained relations exist but they will not be discussed here.

DEFINITION 4. A function of the form

$$P_j(x) = x^{\alpha_j} + A_{j,j-1}x^{\alpha_{j-1}} + \cdots + A_{j,1}x^{\alpha_1} + A_{j,0}x^{\alpha_0}$$

will be called a pseudo-polynomial of degree j. These exponents are positive real numbers and $\alpha_n > \alpha_m$ for $n > m$.

PROPOSITION 1. *The set of all monic polynomials with positive exponents is a subset of the set of all pseudo-polynomials.*

PROOF. Obvious!

THEOREM 3. *Let the nth degree pseudo-polynomial be given by*

$$P_n(x) = x^{\alpha_n} - \sum_{k=0}^{n-1} \prod_{j=0;\, j\neq k}^{n-1} \frac{\alpha_j - \alpha_n}{\alpha_j - \alpha_k} \prod_{j=0}^{n-1} \frac{(\alpha_j + \alpha_k + 1)}{\alpha_j + \alpha_n + 1} x^{\alpha_k};$$

then $\{P_n(x)\}_{n=0}^{\infty}$ *is a set of orthogonal pseudo-polynomials with unity weight function and interval from* 0 *to* 1.

Before proving this theorem we will need two definitions.

DEFINITION 5. Let the notation $\beta_{j,k}$ stand for the integral $\int_0^1 x^{\alpha_j}x^{\alpha_k}\,dx$. From this it is obvious that $\beta_{j,k} = \beta_{k,j}$.

DEFINITION 6. Define the inner product of functions f and g as

$$(f, g) = \int_a^b f(x)g(x)\,dx.$$

Two proofs of Theorem 3 will be given.

FIRST PROOF OF THEOREM 3. Write the pseudo-polynomial $\bar{P}_n(x)$ as the determinant $\bar{P}_n(x) = |x^{\alpha_j}\beta_{j,0}\beta_{j,1} \cdots \beta_{j,n-1}|$ where each entry is a column vector and from top to bottom j takes on the values $n, n-1, \cdots, 0$. This notation will be used below. This pseudo-polynomial has the following property: $\int_0^1 x^{\alpha_j}\bar{P}_n(x)\,dx = 0$ for $j < n$. One can show that $\bar{P}_n(x)$ and the pseudo-polynomial in Theorem 3 differ by only a multiplicative constant.

SECOND PROOF OF THEOREM 3. Take the interval of integration from 0 to 1 and $w(x) = 1$, $f(x) = x^m$, then the inner product is

$$(f, P_k) = \frac{\prod_{j=0}^{k-1}(x_j - \alpha_k)\prod_{j=0}^{k-1}(\alpha_j - m)}{\prod_{j=0}^{k}(m + \alpha_j + 1)\prod_{j=0}^{k-1}(\alpha_j + \alpha_k + 1)}.$$

THEOREM 4. *Let the interval of integration be from* 0 *to* 1 *and the weight function be unity; then the pseudo-polynomials*

$$P_n(x, m) = x^{\alpha_n} - \sum_{k=0}^{n-1} \prod_{j=0;\, j=k}^{n-1} \frac{(\alpha_j - \alpha_n)}{(\alpha_j - \alpha_k)} \prod_{j=0}^{n-m-1} \frac{(\alpha_j + \alpha_k + 1)}{(\alpha_j + \alpha_n + 1)} x^{\alpha_k}$$

are a set of partially-orthogonal pseudo-polynomials of deficiency m and are related to those of Theorem 3. *If* $n-m-1$ *is* <0 *the product is taken to be unity. The nonorthogonal conditions imposed on these pseudo-polynomials are the following*:

$$P_n(1, m) = 0, \qquad \frac{d}{dx}[x^{-\alpha_0}P_n(x, m)]_{x=1} = 0,$$

$$\frac{d}{dx}\left[x^{(\alpha_{m-3}-\alpha_{m-2}+1)}\frac{d}{dx}\left[\cdots\frac{d}{dx}[x^{-\alpha_0}P_n(x, m)]\right]\cdots\right]_{x=1} = 0.$$

PROOF. The following pseudo-polynomial is written as the determinant:

$$\bar{P}_n(x, m) = \left| x^{\alpha_j} 1 \prod_{i=0}^{1}\alpha_{j,i}\cdots\prod_{i=0}^{m-2}\alpha_{j,i}\beta_{j,0}\cdots\beta_{j,k}\right|$$

where each entry is a column vector and from bottom to top, $j=0, i, \cdots, n$, $\alpha_{j,i}=\alpha_j-\alpha_i$, $k=n-m-1$, and $\prod_{i=0}^{L}\alpha_{j,i}=0$ for $L>i$. The set $\{\bar{P}_n(x, m)\}_{n=0}^{\infty}$ is partially orthogonal of deficiency m and has nonorthogonal zeros as prescribed in the theorem. It can be shown that $\bar{P}_n(x, m)$ and $P_n(x, m)$ differ only by a multiplicative constant.

THEOREM 5. *The results of Theorem* 1 *are valid for the pseudo-polynomials of Theorem* 4.

The proof of this is similar to the proof of Theorem 1 so will not be given.

THEOREM 6. *For* $n>m$ *the pseudo-polynomial*

$$R_n(x, m) = x^{\alpha_n} - \sum_{k=m}^{n-1}\prod_{j=m;j\neq k}^{n-1}\frac{(\alpha_j-\alpha_n)}{(\alpha_j-\alpha_k)}\prod_{j=0}^{n-m-1}\frac{(\alpha_k+\alpha_j+1)}{(\alpha_n+\alpha_j+1)}x^{\alpha_k}$$

is a member of a partially-orthogonal set of pseudo-polynomials of deficiency m.

PROOF. Let $f(x)=x^L$, $g(x)=R_n(x, m)$, then the inner product (f, g) can be obtained and is, for $a=0$, $b=1$, and $w(x)=1$,

$$(f, g) = \frac{\prod_{j=m}^{k-1}(\alpha_{m+k}-\alpha_j)\prod_{j=0}^{k-1}(L-\alpha_j)}{\prod_{j=m}^{m+k}(L+\alpha_j+1)\prod_{j=0}^{k-1}(\alpha_{m+k}+\alpha_j+1)}.$$

By the same argument as that of Theorem 1 it can be shown that the results of Theorem 1 are valid for the pseudo-polynomials of Theorem 6.

COROLLARY 3. *The sets of pseudo-polynomials discussed in Theorems* 4 *and* 6 *are both related to the pseudo-polynomials discussed in Theorem* 3.

PROOF. If, in Theorems 4 and 6, $m=0$ the pseudo-polynomials of Theorem 3 are obtained. These three sets of pseudo-polynomials are an example of Theorem 2.

THEOREM 7. *For the pseudo-polynomials*

$$\{R_n(x, m+1)\}_{n=0}^{\infty} \quad \textit{and} \quad \{R_n(x, m)\}_{n=0}^{\infty}$$

the conclusions of Theorem 1 *are valid.*

A different proof than the previous ones will be given here.

PROOF. Assume that a sequence of positive integers $\{b_{n,m+1}\}_{n=0}^{\infty}$, $m \geqq 0$, exists such that

$$R_n(x, m+1) = R_n(x, m) + b_{n,m+1} R_{n-1}(x, m).$$

The $b_{n,m+1}$ can be obtained explicitly as

$$b_{n,m+1} = \frac{(\alpha_m + \alpha_{i+1}) \prod_{k=m+1}^{n-1} (\alpha_k - \alpha_n) \prod_{k=0}^{i-1} (\alpha_{n-1} + \alpha_k + 1)}{\prod_{k=m}^{m-2} (\alpha_k - \alpha_{n-1}) \prod_{k=0}^{i} (\alpha_n + \alpha_k + 1)}.$$

The last section of this paper will discuss polynomials that are essentially the simple Laguerre polynomials. Write $L_n(x) = x^n + A_{n,n-1}x^{n-1} + \cdots + A_{n,1}x + A_{n,0}$ are require $\int_0^\infty e^{-sx} x^j L_k(x)\, dx = 0$ for $j < k$.

DEFINITION 7. Let $f^{(-n)}(x)$ represent

$$\underbrace{\int \cdots \int}_{N \text{ times}} f(x)\, dx^n.$$

The following result is easily obtained so it must be well known. The set $\{L_n^{(-j)}(x)\}_{n=0}^{\infty}$ with all constants of integration zero forms a set of partially-orthogonal polynomials with deficiency j. From Theorem 1, there exist nonzero constants $\{A_{n+j,n+1}\}_{n=0}^{\infty}$ such that $L_{(n+j)}(x, n+1) = L_{(n+j)}(x, n) + A_{n+i,n+1} L_{(n+j-1)}(x, n)$ and $A_{n+j,n+1} = (n+j)/s$. It is obvious that more general integration, such as Lebesgue, could have been used.

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