THE ABSICSSA OF ABSOLUTE SUMMABILITY OF LAPLACE INTEGRALS

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ABSTRACT. With \( A(u) \) of bounded variation over every finite interval of the nonnegative real axis, we write \( C(w) = \int_0^\infty e^{-us} dA(u) \) and (formally)
\[
R(k', w) = \left( \Gamma(k' + 1) \right)^{-1} \int_w^\infty (u - w)^{k'} dA(u) \quad (k' \geq 0).
\]
It is shown that if \( k \) is positive and fractional and if \( e^{-w}\gamma R(k, w) \) is summable \( |C, 0| \) for some \( s' \) whose real part is negative, then \( C(w) \) is summable \( |C, k + e| \) for each \( e > 0 \), where \( s \) is such that its real part is greater than that of \( s' \); if \( k \) is nonnegative and integral the result holds with \( e = 0 \). Together with a 'converse' result, this may be used to show that if the abscissa of \(|C, k|\) summability of \( \int_0^\infty e^{-us} dA(u) \) is negative then it equals
\[
\limsup_{w \to \infty} w^{-1} \log \int_w^\infty |dR(k, u)|
\]
for all \( k \geq 0 \) except one fractional value.

1. If
\[
\Gamma(k + 1)x^{-k}F(a; x) = \int_a^x \frac{f(u) A(u)}{x - u} du = L + o(1)
\]
as \( x \to \infty \) (or is of bounded variation over \([a, \infty)\), with limit \( L \)), where \( A(u) \) is of bounded variation over every finite interval of the nonnegative real axis and the integral is taken in the Riemann-Stieltjes sense, we say that \( F(a; x) \) (i.e., \( F_0(a; x) \)) is summable \( (C, k) \) (or \( |C, k| \)) to \( L \). If now,
\[
(\Gamma(k' + 1))^{-1} \int_w^\infty (u - w)^{k'} dA(u)
\]
is summable \( (C, k) \) (or \(|C, k|\)) to \( L \), we shall say that
\[
R(k', w) = (\Gamma(k' + 1))^{-1} \int_w^\infty (u - w)^{k'} dA(u)
\]
exists in the \((C, k)\) (or \(|C, k|\)) sense and has value \( L \). The notation \( h(x) = L + o(1) |C, k| \) will mean
\[
\int_1^x dh(u) = L - h(1) + o(1) |C, k|.
\]
We shall write $V$ for the class of functions of bounded variation over $[1, \infty)$; $[k]$ for the largest integer less than or equal to $k$, $\langle k \rangle$ for $k - [k]$; $c, c_r, c'_r$ for constants; and

$$C_q(x) = (\Gamma(k + 1))^{-1}\int_0^x (x - u)^ke^{-us}dA(u) \quad (k \geq 0), C(x) = C_0(x),$$

where $s$ is complex and $\text{Re}(s) = \sigma$. We have now,

**Theorem A [6].** If $k$ is positive and fractional, and if $C(w)$ is summable $\langle C, k \rangle$ for some $s$ such that $\sigma < 0$, then $R(k + \delta, w)$ exists in the $\langle C, k \rangle$ sense and

$$e^{-ws}w^{-k}R(k + \delta, w) = o(1) \ |C, 0| \ \text{for each } \delta > 0.$$

**Theorem B.** If $k$ is positive and fractional, and if $R(k, w)$ exists in the $(C, p)$ sense for some $p \geq k$ and satisfies that $e^{-ws}R(k, w)$ is summable $\langle C, 0 \rangle$ for some $s'$ such that $\text{Re}(s') = \sigma' < 0$, then $C(w)$ is summable $\langle C, k + \varepsilon \rangle$ whenever $\sigma > \sigma'$, for every $\varepsilon > 0$.

Theorem B is proved below. For $k$ integral, by [3, Theorem 3] and an easy version of the proof below, we see that Theorems A and B are true with $\varepsilon = 0, \sigma = 0$ respectively. The $(C)$ analogues in all cases are substantially given in [4]. For a general discussion of related results see [8, p. 10].

2. **Proof of Theorem B.** This will follow from

**Lemma 1.** Under the hypotheses of Theorem B, if $0 \leq \varepsilon < 1 - \langle k \rangle$, then for $\sigma > \sigma'$,

$$w^{-k-\varepsilon}C_{k+\varepsilon}(w) + (-1)^{\langle k \rangle}w^{-k-\varepsilon}e^{-ws}U^{\langle k \rangle+\varepsilon}(w) \ \text{is in } V,$$

where

$$U^{(r)}(w) = (\Gamma(r))^{-1}\int_{w-1}^w (w - u)^{-r-1}R([k], u) \ du \quad (r > 0).$$

**Lemma 2.** Under the hypotheses of Theorem B,

$$e^{-ws}U^{\langle k \rangle+\varepsilon}(w) \ \text{is in } V \ \text{for each } \varepsilon > 0.$$

**Proof of Lemma 1.** Case $\varepsilon = 0$. We may take $A$ continuous on the right. Then by [5, Theorem 2], and [4, Lemma 2, Corollary],

$$R([k], u) = (\Gamma(1 - \langle k \rangle))^{-1}\left(\int_u^{u+1} + \int_{u+1}^\infty\right)(t - u)^{-\langle k \rangle}dR(k, t);$$

we see that the integral of $R([k], u)$ from $w$ to $\infty$ is convergent, and hence by [5, p. 236 line $-2$],

$$R([k] + 1, u) \ \text{exists in the } (C, p) \ \text{sense.}$$
We now write

(2)  
(a) \( R(k, t) = e^{st}f(t) \);  
(b) \( R([k] + 1, t) = e^{st}g(t) \).

Then by hypothesis, \( f(t) \) is in \( V \). By [4, Lemma 2],

\[
R([k] + 1, w) = (\Gamma(1 - \langle k \rangle))^{-1} \int_w^\infty (t - w)^{-\langle k \rangle} R(k, t) \, dt,
\]

the integral being convergent by our hypothesis. Inserting (2)(a), then putting \( t = w + x \) and applying [6, Lemma 2], we see that \( g(w) \) is in \( V \). By [4, (64)-(70)] we have

\[
C_k(w) = \sum_{r=0}^{[k]} c_r w^{k-r} + \sum_{r=0}^{[k]+1} c'_r I_r = A + B,
\]
say, where

\[
I_r = \int_0^w R([k], u)(w - u)^{k-r} e^{-us} \, du.
\]

Writing \( R([k], u) \) as the derivative of \( -e^{st}g(u) \) (by (1) and [4, Lemma 2, Corollary]) we now obtain

(5) \( w^{-k}I_r = -w^{-r}(s'N_1(r, w) + N_2(r, w)) \quad (r = 0, 1, \cdots, [k]) \)

where

\[
N_1(r, w) = \int_0^w (1 - u/w)^{k-r} e^{-u(s-s')}g(u) \, du
\]

and \( N_2(r, w) \) is the same integral with \( g \) replaced by \( g' \). Also

\[
I_{[k]+1} = \left( \int_0^{w-1} + \int_{w-1}^w \right) R([k], u)(w - u)^{\langle k \rangle - 1}e^{-us} \, du = P + Q,
\]
say. But

(6) \( P = -w^{\langle k \rangle - 1}(s'N_1([k] + 1, w - 1) + N_2([k] + 1, w - 1)) \)

and \( Q \) may be expressed as

\[
e^{-ws} \int_{w-1}^w R([k], u)(w - u)^{\langle k \rangle - 1} \, du
\]

\[
+ \int_{w-1}^w R([k], u)(w - u)^{\langle k \rangle - 1}(e^{-us} - e^{-ws}) \, du = Q_1 + Q_2,
\]
say. An integration by parts of \( Q_2 \), followed by the insertion of (2)(b)
and then the substitution \( u = w - x \), gives

\[
Q_2 = (1 - e^{-s})e^{-(w-1)(s-s')}g(w - 1) - s \int_0^1 x^{(k)-1}e^{-(w-x)(s-s')}g(w - x) \, dx
- ((k) - 1) \int_0^1 x^{(k)-2}(1 - e^{-s})e^{-(w-z)(s-s')}g(w - x) \, dx.
\]

Applying [4, Lemma 2] to the last two integrals, and also to those in (6) and (5), shows that \( Q_2 \) and \( P \) are in \( V \), and that \( w^{-k}I_r \) is in \( V \) \((r = 0, 1, \cdots, [k])\). Since \( Q_1 \) is just \( \Gamma((k))e^{-wU^{(k)}}(w) \) and \((\text{in (4)})\)
\[
c'_k+1 = (-1)^{k+1}(\Gamma((k)))^{-1},
\]
this completes the proof.

Case \( e > 0 \). By (1) and [1, p. 300], \( R(k+\epsilon, w) \) certainly exists in the \((C, \rho)\) sense, and thus by [4, Lemma 2], (3) holds with \( R([k]+1, w) \) and \(-((k)) \) replaced by \( R(k+\epsilon, w) \) and \( \epsilon - 1 \) respectively. Hence \( e^{-w}R(k+\epsilon, w) \) is in \( V \). The required result now follows from the previous case with \( k \) replaced by \( k+\epsilon \).

**Proof of Lemma 2.** We write \( S(u, w) = R([k]+1, w) - R([k]+1, u) \).
Then by (3) and (2)(a), 
\[
\int_u^w (t-u)^{-((k))}e^{ts}f(t) \, dt + \int_w^\infty \{(t-u)^{-((k))} - (t-w)^{-((k))}\}e^{ts}f(t) \, dt.
\]

Now by an integration by parts, \( e^{-w}U^{(k)+\epsilon}(w) \) has value
\[
c_1e^{-w}S(w - 1, w) + c_2e^{-w} \int_{w-1}^w S(u, w)(w - u)^{(k)+\epsilon-2} \, du = G + H,
\]
say. By (2)(b), \( G \) is in \( V \). We now insert (8) in \( H \), obtaining \( H_1 + H_2 \), say. In \( H_1 \) we put \( t = w - y \) and then \( u = w - x \); in \( H_2 \) we put \( t = w + v \), \( u = w - x \). Applying [6, Lemma 2], we see that \( H_1 \) and \( H_2 \) are in \( V \). This completes the proof.

3. **The abscissa \( \overline{d}_k \).** We shall write \( \overline{d}_k \) for the infimum of the set of \( \sigma \)'s for which \( C(w) \) is summable \([C, k]\), and \( \overline{k} \) for the infimum of the set of \( k \)'s for which \( \overline{d}_k \) is less than infinity. It is known (see [9], [7], [2, Lemma 13]) that \( \overline{d}_k \) is continuous for \( k > \overline{k} \), the value \(-\infty \) being allowed. We have now, as a deduction from Theorems A and B (compare [4, pp. 470, 475]):

**Theorem A*.** If \( k \) is positive and fractional, \( k > \overline{k} \) and \( C(w) \) is summable \([C, k]\) for some \( s \) such that \( s < 0 \), then \( R(k, w) \) exists in the \([C, k]\) sense and \( e^{-w}R(k, w) = o(1) [C, 0] \) for \( \sigma' > \sigma \).

**Theorem B*.** If \( k \) is positive and fractional, \( k > \overline{k} \) and if \( R(k, w) \) exists in the \((C, p)\) sense for some \( p \geq k \), and
\[
e^{-w}R(k, w) \text{ is summable } [C, 0] \text{ for some } s' \text{ such that } \sigma' < 0,
\]
then \( C(w) \) is summable \([C, k]\) for every \( \sigma > \sigma' \).
We observe that the conclusion of Theorem A* implies
\begin{equation}
\int_{w}^{\infty} |d_w R(k, w)| = O(\omega^{\sigma'}) \quad (\sigma' > \sigma)
\end{equation}
and it is not difficult to see that (10), together with the condition
\begin{equation}
R(k, w) \to 0 \quad \text{as} \quad w \to \infty
\end{equation}
implies the hypothesis (9) of Theorem B*, with a slightly larger \( \sigma' \). Since certainly \( R(k, w) \) exists in the \((C, k)\) sense (by [1, p. 300]) and also (11) holds (by Theorem A*) if we assume that \( \sigma_k \) is negative, we obtain the following formula for \( \sigma_k \) in the case \( k \) fractional (see [4, p. 463]), a similar argument yielding the case \( k \) integral:

**Theorem C.** Let \( \sigma_k \) be negative. Then if \( k=0, 1, \cdots, \) or if \( k \) is fractional and \( k > k_0 \),
\begin{equation}
\sigma_k = \limsup_{w \to \infty} \frac{1}{w} \log \int_{w}^{\infty} |dR(k, u)|.
\end{equation}
If \( k \) is fractional and \( k = k_0 \), it is possible for the formula to fail.

For the second part we choose the function \( A(u) \) used in the proof of [6, Theorem A*]; then the right side of (12) is infinity.

**Bibliography**


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