A SMALL BOUNDARY FOR \( H^\infty \) ON THE POLYDISC

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Abstract. Let \( \Delta^n \) be the unit polydisc in \( \mathbb{C}^n \) and let \( T^n \) be its distinguished boundary. It is shown that for \( n \geq 2 \) there is a nowhere dense subset of the maximal ideal space of \( L^\infty (T^n) \) which defines a closed boundary for \( H^\infty (\Delta^n) \).

Let \( H^\infty (\Delta^n) \) be the Banach algebra of bounded holomorphic functions on the open unit polydisc \( \Delta^n = \{ |z_i| < 1, 1 \leq i \leq n \} \) in \( \mathbb{C}^n \). Denote by \( T^n = \{ |z_i| = 1, 1 \leq i \leq n \} \) the distinguished boundary of \( \Delta^n \), and let \( \sigma \) be the area measure on \( T^n \). By taking radial limits, each \( f \in H^\infty (\Delta^n) \) defines almost everywhere on \( T^n \) a function \( f^* \in L^\infty (T^n, \sigma) \); \( H^\infty (\Delta^n) \) can thus be identified with a closed subalgebra of \( L^\infty (\sigma) \) (see, for example, W. Rudin [3]).

Let \( \mathcal{M}_n \) and \( X_n \) be the maximal ideal spaces of \( H^\infty (\Delta^n) \) and \( L^\infty (\sigma) \) respectively. A closed boundary for \( H^\infty (\Delta^n) \) is a closed subset \( \Gamma \) of \( \mathcal{M}_n \), such that \( \| f \| = \sup \{ |\gamma (f)| : \gamma \in \Gamma \} \) for all \( f \in H^\infty (\Delta^n) \). The above identification gives rise to a continuous map \( \tau : X_n \to \mathcal{M}_n \) defined by

\[
\tau (\varphi) (f) = \varphi (f^*) \quad \text{for} \ \varphi \in X_n \ \text{and} \ f \in H^\infty (\Delta^n),
\]

whose image \( \tau (X_n) \) is a closed boundary for \( H^\infty (\Delta^n) \).

It is known that for \( n = 1 \), the map \( \tau : X_1 \to \mathcal{M}_1 \) is actually a homeomorphism from \( X_1 \) onto the Shilov boundary (i.e. the smallest closed boundary) of \( H^\infty (\Delta^1) \) (see K. Hoffman [2, p. 174]).

The purpose of this note is to show that the corresponding result is no longer true in higher dimensions. This will be accomplished by constructing a "very small" subset of \( X_n \) which maps onto a boundary. The precise statement is as follows.

Theorem. Let \( n \geq 2 \). There is a closed, nowhere dense subset \( \beta \) of \( X_n \) with measure \( \delta (\beta) = 0 \) such that \( \tau (\beta) \) is a closed boundary for \( H^\infty (\Delta^n) \).

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Here $\hat{\delta}$ denotes the unique regular Borel measure on $X_n$ which satisfies
\[
\int_{\Gamma_n} g \, d\sigma = \int_{X_n} \hat{g} \, d\hat{\delta}
\]
for all $g \in L^\infty(\sigma)$ ($\hat{g}$ denotes the Gelfand transform of $g$).

**Remarks.** An analogous result is valid for the unit ball in $C^n$. It seems to be an open problem to characterize the Shilov boundary of $H^\infty(\Delta^n)$. The above result does not give information on whether $\tau$ is one-to-one or $\tau(X_n)$ is the Shilov boundary. However, it shows that at most one of these statements can be true.

**Proof.** For simplicity, we will only consider the case $n=2$. The general case can be handled similarly. The construction of $\beta$ will be done in three steps. First, one constructs a suitable sequence $\{E_k\}_{k=1}^\infty$ of subsets of $T^2$. Then one sets $U_k = \{\varphi \in X_2 : \varphi(\chi_{E_k}) = 1\}$, and one shows that $\tau(U_k)$ is a boundary. Finally, one verifies that $\beta = \bigcap_{k=1}^\infty U_k$ has all the desired properties.

**Step 1.** It will be convenient to parametrize the torus $T^2$ in such a way that the circles $\{e^{i\omega\theta} : 0 \leq \theta \leq 2\pi\}$ for $\omega \in T^2$ correspond to lines parallel to a coordinate axis. Thus, let $Q = [0, \pi] \times [0, 2\pi] \subset \mathbb{R}^2$, and define the continuous map $p : Q \to T^2$ by
\[
p(x, y) = (e^{i(x+\pi y)}, e^{i(-x+\pi y)}).
\]
$p$ is onto, $p(\partial Q)$ has measure 0, and $p$ is a diffeomorphism from the interior of $Q$ onto $T^2 - p(\partial Q)$. If $w_0 = p(x_0, y_0)$ is in $T^2$, then
\[
\{e^{i\omega\theta} : 0 \leq \theta \leq 2\pi\} = \rho(\{(x_0, y) : 0 \leq y \leq 2\pi\}).
\]

Let $(r_j)_{j=1}^\infty$ be an enumeration of the rational numbers in $(0, \pi)$. Fix a positive integer $k$, let $I_j^k = \{x \in (0, \pi) : |x - r_j| < (1/2k)(1/2^{j+1})\}$ for $j = 1, 2, \ldots$, and set $I_k = \bigcup_{j=1}^\infty I_j^k$. One verifies easily that $E_k = \rho(I_k \times [0, 2\pi])$ is an open, dense subset of $T^2$ with $\sigma(E_k) < 1/k$, and from (*) it follows that
\[
\{e^{i\omega\theta} : 0 \leq \theta \leq 2\pi\} \subset E_k.
\]

**Step 2.** Let $U_k = \{\varphi \in X_2 : \varphi(\chi_{E_k}) = 1\}$, where $\chi_{E_k}$ is the characteristic function of $E_k$. $U_k$ is a closed open subset of $X_2$ with $\delta(U_k) = \sigma(E_k) < 1/k$ (see T. W. Gamelin [1, Chapter I]). In order to show that the closed set $\tau(U_k) \subset M_2$ is a boundary, we define, given $f \in H^\infty(\Delta^2)$, a function $G_f : T^2 \to \mathbb{R}$ by
\[
G_f(w) = \sup_{\lambda \in \Delta} |f(\lambda w)| = \text{ess sup} |f^*(\omega w)|
\]
for $w \in T^2$. One shows easily that $G_f$ is lower semicontinuous (Rudin [3, Theorem 3.5.2]).
Now let \( f \in H^\omega(\Delta^2) \), and assume that \( \sup_{t \in U_k} |f| = \text{ess sup}_{t \in U_k} |f^*| \leq 1 \). Denote by \( m_i \) Lebesgue measure in \( \mathbb{R}^i \), for \( i = 1, 2 \). The function \( g = |f^*| \circ \rho \) is in \( L^\infty(Q, m_2) \), and \( g \leq 1 \) \( m_2 \)-a.e. on \( I_k \times [0, 2\pi] \). This implies that for \( m_1 \)-almost all \( x \in I_k \) one has \( \text{ess sup}_{0 \leq t \leq 2\pi} g(x, y) \leq 1 \), and hence, by (*) and (**) \[
G_f(w) = \text{ess sup}_{|z| = 1} |f^*(zw)| \leq 1 \quad \sigma\text{-a.e. on } E_k.
\]
Since \( G_f \) is lower semicontinuous, and since \( E_k \) is open and dense in \( T_2 \), it follows that \( G_f(w) \leq 1 \) for all \( w \in T_2 \), which implies \( \|f\| \leq 1 \). Thus, \( \tau(U_k) \) is a closed boundary for \( H^\omega(\Delta^2) \).

**Step 3.** The construction of \( E_k \) shows that \( E_k \subseteq E_{k+1} \), and hence also \( U_k \subseteq U_{k+1} \) for \( k = 1, 2, \ldots \). A standard compactness argument then shows that \( \beta = \bigcap_{k=1}^{\infty} U_k \) is nonempty and that \( \tau(\beta) = \bigcap_{k=1}^{\infty} \tau(U_k) \).

Therefore, being an intersection of closed boundaries, \( \tau(\beta) \) is a closed boundary as well. Clearly \( \delta(\beta) = 0 \), and since the closed support of \( \delta \) is \( X_2 \) \([1, 1.9.2]\), \( \beta \) has no interior. Thus \( \beta \) has all the required properties, and the theorem is proved.

**Note.** It was remarked by J. P. Rosay that a simple modification of the above argument shows that \( \tau : X_2 \to M_2 \) is not one-to-one. One replaces the set \( I_k \subseteq (0, \pi) \) in Step 1 by a measurable set \( A \subseteq (0, \pi) \) satisfying \( 0 < m_1(A \cap V) < m_1(V) \) for all open sets \( V \subseteq (0, \pi) \). As in Step 2, it follows that \( E = \rho(A \times [0, 2\pi]) \) and \( T^2 - E \) define two disjoint sets in \( X_2 \) which map onto a boundary for \( H^\omega(\Delta^2) \).

ADDED IN PROOF: J. P. Rosay observed that the results of this paper imply that \( \tau(X_2) \) is strictly larger than the Shilov boundary. In fact, let \( E \) be any of the sets constructed in Step 1, and let \( F \subset T^2 - E \) be a closed set with positive measure. It follows from [3, 3.5.3] that there is \( f \in H^\omega(\Delta^2) \) with \( |f^*| = 1 \) on \( F \) and \( |f^*| = 2 \) on \( F^c \). Thus \( |f| = 2 \) on the Shilov boundary, but \( |f| \neq 2 \) on \( \tau(X_2) \).

**REFERENCES**


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