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AN EMBEDDING CHARACTERIZATION OF ALMOST REALCOMPACT SPACES

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ABSTRACT. Any Hausdorff space X can be embedded into the product space $\Pi\{R_f^+ : f \in C_+(X)\}$, where R^+ is the set of all non-negative reals with the topology consisting of R^+ and all sets of the form $\{x \in R^+ : x < a\}$, $a \in R$, and $C_+(X)$ is the set of all continuous functions from X to R^+. Almost realcompact Hausdorff spaces are characterized as maximal Hausdorff subspaces in their closures in the product.

1. **Introduction.** All spaces are assumed to be separated. Recall that a space P is said to be almost compact if every maximal open filter has a cluster point, and almost realcompact if every maximal open filter with the C.C.I.P. has a cluster point. A filter has the C.C.I.P. if every countable subcollection has a cluster point. We refer to [**4**] for general properties of almost realcompact spaces.

Every realcompact space is almost realcompact, and a completely regular almost realcompact space is realcompact whenever it is either normal and countably paracompact [**4**, Theorem 11] or extremally disconnected (this follows easily from [**4**, Theorem 13]). R. Blair and S. Mrówka noticed that it follows from Mrówka's example of the non-realcompact union of two closed realcompact spaces [**7**] that the proper image of a realcompact space need not be realcompact, and since the class of all almost realcompact spaces is closed under proper mappings, this yields an example of an almost realcompact completely regular space that is not realcompact. Almost realcompact regular spaces are just the proper images of realcompact spaces because every regular space is a proper image of an extremally disconnected space namely its projective resolution [**2**]. We do not know any similar characterization of separated almost realcompact spaces.

Realcompact spaces are just the homeomorphs of closed subspaces of the powers of R (in the terminology of Engelking-Mrówka [**1**], realcompact spaces are just the R-compact spaces). Our main result (Theorem

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4 below) is a similar characterization of almost realcompact spaces. Namely every almost realcompact space is just a maximal separated subspace in its closure in some power of nonnegative reals endowed with the topology giving upper semicontinuity. We do not know whether almost realcompact completely regular spaces can be defined as E-compact spaces for some E. On the other hand, it is clear that almost realcompact spaces cannot be characterized as E-compact spaces because a closed subspace of an almost realcompact space need not be almost realcompact (unless the space is regular).

Compact spaces are characterized as homeomorphs of closed subspaces of the powers of the closed unit interval. Here we characterize almost compact spaces (Theorem 5 below) as maximal separated subspaces in their closures in some powers of unit interval considered as a subspace of the nonnegative reals with the topology described above. Given a space E, not necessarily separated, one can define an almost E-compact space as a space P such that P is a maximal separated subset in its closure in some power of E. Thus E-compact spaces are almost E-compact. The first author is presently engaged in writing a paper on a general theory of almost E-compact spaces with particular attention to the spaces E that are complete with respect to a collection of coverings.

2. **Embedding characterization.** The Katětov extension **[5]** of a space X is an almost compact space κX such that X is dense in κX, $\kappa X - X$ is a discrete closed subset of κX, and the open neighborhoods of any point of $\kappa X - X$ intersect X in a maximal open filter. We shall think of the points of $\kappa X - X$ as maximal nonconvergent open filters on X. Thus, if $p \in \kappa X - X$, then $\{p\} \cup U$, for $U \in p$, forms a local base at p. The subspace ρX of κX consisting of all $x \in X$ and all $p \in \kappa X - X$ with the C.C.I.P. in X is the smallest almost realcompact subspace of κX that contains X, see **[6]**.

We denote by R^+ the set of all nonnegative reals with the topology consisting of R^+ and all sets of the form $\{x \in R^+ : x < a\}$, $a \in R$. Thus $f: X \to R^+$ is continuous iff $f: X \to R$ is upper semicontinuous. We denote by $C_+(X)$ the set of all continuous $f: X \to R^+$.

Let α be the collection of all countable open coverings of X. Let β be the collection of all families $B(f) = \{\{x : f(x) < n\} | n = 1, 2, \cdots\}$ for each $f \in C_+(X)$.

The following definition and lemma can be found in **[3]**, **[4]**.

DEFINITION 1. Let $\mathscr{U}$ be an open ultrafilter on X. Then $\mathscr{U}$ is said to be α-Cauchy provided that for each $\mathscr{C} \in \alpha$, there exist $C \in \mathscr{C}$ and $U \in \mathscr{U}$ with $U \subset C$. Equivalently, $\mathscr{C} \cap \mathscr{U} \neq 0$ for each $\mathscr{C} \in \alpha$. Similarly, $\mathscr{U}$ is β-Cauchy provided that, for each $f \in C_+(X)$, there exist $n \in N$ and $U \in \mathscr{U}$ such that $f(x) < n$ for all $x \in U$ (i.e. f is bounded on some $U \in \mathscr{U}$).

It is clear that if γ is the collection of all nested countable open coverings of X, then $\mathscr{U}$ is α-Cauchy iff $\mathscr{U}$ is γ-Cauchy.

LEMMA [4]. *An open ultrafilter $\mathscr{U}$ is α-Cauchy iff $\mathscr{U}$ has the C.C.I.P. on X.*

THEOREM 1. *An open ultrafilter $\mathscr{U}$ in X has the C.C.I.P. iff for each $f\in C_+(X)$, f is bounded on some $U\in\mathscr{U}$.*

Since every α-Cauchy family $\mathscr{U}$ is clearly β-Cauchy, in view of the above lemma, Theorem 1 is an immediate consequence of the following lemma.

LEMMA 1. *If an open ultrafilter $\mathscr{U}$ in X is β-Cauchy, then $\mathscr{U}$ is α-Cauchy.*

PROOF. Suppose there exists $\mathscr{C}=\{C_n:n=1,2,\cdots\}\in\alpha$ such that $U\not\subset C_n$ for all $U\in\mathscr{U}$ and n. We may assume $\mathscr{C}$ is nested, i.e. $C_1\subset C_2\subset\cdots\subset C_n\subset\cdots$.

For each $x\in X$, let $f(x)$ be defined as $\min\{k:x\in C_k\}=n$. If $\varepsilon>0$ is given, then C_n is a neighborhood of x such that if $y\in C_n$, then $f(y)\leqq n<n+\varepsilon=f(x)+\varepsilon$. Thus $f\in C_+(X)$. Obviously, f is bounded on no U in $\mathscr{U}$, which contradicts the fact that $\mathscr{U}$ is β-Cauchy. The proof is complete.

For the next theorem, we need the following little surprising lemma.

LEMMA 2. *If $f\in C_+(X)$ and if $\mathscr{U}$ is a maximal open filter in X such that f is bounded on some element of $\mathscr{U}$, then the image $f(\mathscr{U})$ of $\mathscr{U}$ converges to a point of R^+ in the usual topology of R.*

PROOF. There exists a half open interval $[a,b)\subset R_+$ and a $U\in\mathscr{U}$ such that $f(U)\subset[a,b)$. Let $\varepsilon>0$ be given. Divide $[a,b)$ into $n+1$ parts each with length at most ε, say $a<a+\varepsilon<\cdots<a+k\varepsilon<\cdots<a+n\varepsilon<b$. Let k be such that $f^{-1}(\{x:x<a+k\varepsilon\})\notin\mathscr{U}$ but $f^{-1}(\{x:x<a+(k+1)\varepsilon\})\in\mathscr{U}$. Thus there exists $V\in\mathscr{U}$ such that $V\cap f^{-1}(\{x:x<a+k\varepsilon\})$ is empty. Let $G=V\cap f^{-1}(\{x:x<a+(k+1)\varepsilon\})$. Clearly $\operatorname{diam}(f(G))\leqq\varepsilon$. Consequently $f(\mathscr{U})$ converges in R^+ in the usual topology.

THEOREM 2. *For any space X, $\rho X-X$ is equal to the set A of all $p\in\kappa X-X$ such that every $f\in C_+(X)$ extends to a $\bar f\in C_+(X\cup\{p\})$.*

PROOF. If $p\in\rho X-X$, then p corresponds to a maximal open filter $\mathscr{P}$ in X with the C.C.I.P. By Theorem 1 and Lemma 2, we have $f(\mathscr{P})$ converges to a point $r\in R^+$ for each $f\in C_+(X)$. Define $\bar f(\mathscr{P})=r$ and $\bar f(x)=f(x)$ for each $x\in X$. Given $\varepsilon>0$, since $f(\mathscr{P})$ converges to r, there exists a $G\in\mathscr{P}$ such that $f(G)\subset(r-\varepsilon,r+\varepsilon)$. Clearly $W=G\cup\{\mathscr{P}\}$ will be a neighborhood of $\mathscr{P}$ in κX such that $\bar f(W)\subset(r-\varepsilon,r+\varepsilon)$. Therefore $\bar f\in C_+(X\cup\{p\})$, and consequently $(\rho X-X)\subset A$.

Conversely, if $p\in\kappa X-\rho X$, then p is a nonconvergent maximal open filter in X which does not have the C.C.I.P. By Theorem 1, there exists an

$f \in C_+(X)$ which is unbounded on every $G \in \mathscr{P}$. Clearly f cannot be extended to an $\bar{f} \in C_+(X \cup \{\mathscr{P}\})$.

THEOREM 3. *A space X is almost realcompact iff for each $p \in \kappa X - X$, there exists an $f \in C_+(X)$ such that f cannot be extended to an $\bar{f} \in C_+(X \cup \{p\})$.*

PROOF. Since X is almost real compact iff $X = \rho X$, this theorem is an immediate consequence of Theorem 2.

We will denote the product space $\Pi\{R_f^+ : f \in C_+(X)\}$ by R^{C_+}. The continuous map ϕ from X into R^{C_+} is defined by $(\phi(x))_f = f(x)$ for each $f \in C_+(X)$.

COROLLARY. *X is almost realcompact iff for each $\mathscr{P} \in \kappa X - X$, the continuous map ϕ has no continuous extension from $X \cup \{\mathscr{P}\}$ into R^{C_+}.*

Let us denote by χ_A the characteristic function of A.

We will omit the proof of the following simple lemma.

LEMMA. *Let $A \subset X$. Then $\chi_A \in C_+(X)$ iff A is closed.*

REMARKS. 1. Since the continuous characteristic functions distinguish points and closed sets in X (by the lemma above), it is clear that ϕ is an embedding.

2. It is clear that no nonempty closed set in R^{C_+} is separated; therefore $\phi[X]$ is not closed in R^{C_+} if X is not empty.

3. Let $p \in \kappa X - X$. Suppose that for each $f \in C_+(X)$, there exists a $g \in C_+(X \cup \{p\})$ such that $g|X = f$; then the minimum extension $\bar{f}$ of f defined by $\bar{f}(x) = f(x)$ for each $x \in X$ and $\bar{f}(p) = \limsup_{x \to p} f(x)$ exists, and $\bar{f} \in C_+(X \cup \{p\})$. Let $\bar{\phi}$ be the map from $X \cup \{p\}$ into R^{C_+} defined by $(\bar{\phi}(t))_f = \bar{f}(t)$ for each $t \in X \cup \{p\}$. Now if A is closed in $X \cup \{p\}$ and $p \notin A$, then there exists an open neighborhood $W = G \cup \{p\}$ of p, where G is a member of the maximal open filter corresponding to p, such that $W \subset (X \cup \{p\}) - A$. Then the minimum extension $\bar{\chi}_A$ of χ_A has the property that $\bar{\chi}_A(p) = 0$. Thus the continuous characteristic functions on $X \cup \{p\}$ distinguish points and closed sets and consequently $\bar{\phi}$ is an embedding.

THEOREM 4. *A space X is almost realcompact iff $\phi[X]$ is a maximal separated subspace of $\phi[X]^-$, where the closure is taken in R^{C_+}.*

PROOF. Necessity. Suppose there exists $p \in \phi[X]^- - \phi[X]$ such that $\phi[X] \cup \{p\}$ is separated. If $\mathscr{U}(p)$ is the collection of all open neighborhoods of p in R^{C_+}, then $\mathscr{G} = \mathscr{U}(p) \cap \phi[X]$ is an open filter in $\phi[X]$. Consequently $\mathscr{U} = \{\phi^{-1}(G) : G \in \mathscr{G}\}$ is an open filter in X. Let $\mathscr{U}'$ be a maximal open filter in X containing $\mathscr{U}$. Clearly $\bigcap \bar{\mathscr{U}}' = \varnothing$; thus $\mathscr{U}' \in \kappa X - X$. Define $\bar{\phi}(\mathscr{U}') = p$ and $\bar{\phi}(x) = \phi(x)$ for all $x \in X$. We will prove that $\bar{\phi}$ is a continuous extension of ϕ from $X \cup \{\mathscr{U}'\}$ into R^{C_+}, which will contradict the corollary of Theorem 3. Clearly $\bar{\phi}$ is continuous at each point x in X since X is open in $X\cup\{\mathscr{U}'\}$. Now if G is open in R^{C_+} containing p, then $G\cap\phi[X]\in\mathscr{G}$ and consequently $W=\phi^{-1}(G\cap\phi[X])\in\mathscr{U}'$. Thus $W\cup\{\mathscr{U}'\}$ is an open neighborhood of $\mathscr{U}'$ in $X\cup\{\mathscr{U}'\}$ such that $\bar{\phi}(W\cup\{\mathscr{U}'\})\subset(G\cap\phi[X])\cup\{p\}$. The proof is complete.

To show the sufficiency, suppose X is not almost realcompact. Then there exists a $p\in\kappa X-X$ such that every $f\in C_+(X)$ can be extended to an $g\in C_+(X\cup\{p\})$. Since the map $\bar{\phi}$ (as described in Remark 3 above) from $X\cup\{p\}$ into R^{C_+} is an embedding, $\bar{\phi}(X\cup\{p\})=\phi[X]\cup\{\bar{\phi}(p)\}$ is a separated subspace of $\phi[X]^-$ that properly contains $\phi[X]$.

Denote by I^+ the closed interval [0, 1] considered as a subspace of R^+, and by $B_+(X)$ the set of all continuous functions of X into I^+. The product space $\Pi\{I_f^+ : f\in B_+(X)\}$ will be denoted by I^{B_+}. For each $p\in\kappa X-X$, p corresponds to a maximal open filter $\mathscr{U}_p$ in X. By Lemma 2, $f(\mathscr{U}_p)$ converges to a point r in I^+ for each $f\in B_+(X)$. Consequently each $f\in B_+(X)$ extends to a $g\in B_+(X\cup\{p\})$, given by $g(p)=r$. If $\bar{f}$ is the minimum extension of f as described in Remark 3, then the same argument shows that $\bar{\psi}:X\cup\{p\}\to I^{B_+}$ defined by $(\bar{\psi}(x))_f=\bar{f}(x)$ is an embedding. The proof of the following theorem is thus quite similar to that of Theorem 4. We will leave the details to the reader.

THEOREM 5. *A space X is almost compact iff $\psi[X]$ is a maximal separated subspace of $\psi[X]^-$, where the closure is taken in I^{B_+}.*

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