THE NUMBER OF ISOTYPE AND \( \lambda \)-PURE SUBGROUPS
OF AN ABELIAN \( p \)-GROUP

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ABSTRACT. The number of isotype subgroups of an abelian \( p \)-group \( G \) is determined. This solves a recent problem of Fuchs. Actually, we accomplish slightly more. Define a subgroup \( H \) of an abelian \( p \)-group \( G \) to be an \( \lambda \)-pure subgroup of \( G \) if, for some ordinal \( \lambda \), \( H \) is \( p^\lambda \)-pure in \( G \) and \( p^\lambda H \) is divisible. We compute the number of \( \lambda \)-pure subgroups of \( G \) and show that the number of \( \lambda \)-pure subgroups and the number of isotype subgroups of \( G \) coincide. Our final result deals with the number of nonisomorphic isotype subgroups of \( G \) when \( G \) is a direct sum of countable groups.

In this paper all groups are abelian, and “abelian group” is shortened to “group”. For the most part, our notation and terminology agree with [2]. However, for the convenience of the reader we list the meaning of the following frequently used symbols and terms.

\(+\) : direct sum.
\(\sum\) : direct sum (of an infinite family).
\(|X|\) : cardinality of \( X \).
\(l(G)\) : the length of a reduced \( p \)-group \( G \).
\(p^\alpha G[p]\) : \((p^\alpha G)[p]\).
c : cardinality of the continuum, \(2^{\aleph_0}\).
\(\omega, \Omega\) : the first infinite ordinal and the first uncountable ordinal, respectively.

isotype subgroup: \( H \) is an isotype subgroup of the \( p \)-group \( G \) if
\(p^\alpha G \cap H = p^\alpha H\) for every ordinal \( \alpha \).

\(\lambda\)-pure subgroup: \( H \) is an \( \lambda \)-pure subgroup of the \( p \)-group \( G \) if, for some ordinal \( \lambda \), \( H \) is \( p^\lambda \)-pure in \( G \) and \( p^\lambda H \) is divisible; recall that \( H \) is \( p^\lambda \)-pure in \( G \) if \( H \rightarrow G \rightarrow G/H \) belongs to \( p^\lambda \text{Ext}(G/H, H) \).

subsocle: a subsocle of a \( p \)-group \( G \) is any subgroup of \( G[p] \).

In [2] Fuchs raised the question: What is the cardinality of the set of all pure subgroups of a group \( G \)? The answer to this problem was provided by the combination of two papers, one by Boyer [1] and the other by Hill [4]. In his new book [3], Fuchs has presented as Problem 20(a):

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What is the cardinality of the set of [nonisomorphic] isotype subgroups of a \( p \)-group \( G \)? We can answer completely the primary question: What is the cardinality of the set of isotype subgroups of \( G \)? We suggest that the theory of abelian groups may not yet be advanced enough to solve, in a meaningful way, the secondary problem of determining for all \( p \)-groups \( G \) the number of nonisomorphic isotype subgroups. However, this is certainly possible for many special cases of \( G \). For example, if \( G \) is a \( p \)-group of cardinality \( c \) without elements of infinite height having a countable basic subgroup, then \( G \) has \( 2^c \) isomorphically distinct isotype subgroups according to Theorem 2.3 of [7]. In the present paper (Theorem 3), we determine the number of isomorphically distinct isotype subgroups of a \( p \)-group \( G \) of countable length that can be written as a direct sum of less than \( \aleph_0 \) countable groups.

**Lemma 1.** Let \( G \) be a \( p \)-group. Suppose, for an ordinal \( \alpha \), that \( m_\alpha < m \geq \aleph_0 \) where \( m_\alpha = |G[p]/ p^\alpha G[p]| \) and \( m = |G[p]| \). Let \( S \) be a subsocle of \( G \) such that \( S \cap p^\alpha G[p] = 0 \). If \( T \) is a subsocle of \( G \) such that \( |T| < m \) and such that \( T \cap S = 0 \), then there exists \( S^* \subseteq G[p] \) such that:

1. \( S^* \supseteq S \),
2. \( S^* + p^\alpha G[p] = G[p] \),
3. \( S^* \cap T = 0 \).

**Proof.** First, choose \( R \supseteq S \) such that \( R \) is maximal in \( G[p] \) with respect to \( R \cap p^\alpha G[p] = 0 \). Then \( S^* = R \) satisfies conditions (1) and (2). If \( R \cap T = 0 \), the proof is finished. Thus we may assume that \( R \cap T \neq 0 \). Write \( p^\alpha G[p] = \sum_{j \in J} \{y_j\} \), and note that \( |J| = m \). Since \( G[p] = R + \sum_{j \in J} \{y_j\} \), we have, for a suitable choice of the basis \( \{y_j\} \), that \( \langle R, T \rangle = R + \sum_{j \in K} \{y_j\} \) for some subset \( K \) of \( J \). Also, \( |K| < |J| = m \) since \( T \) has cardinality less than \( m \). Write \( R = S + W \), and set \( W = \sum_{i \in I} \{x_i\} \). Observe that \( |J - K| = m > m_\alpha \geq |I| \), so there exists a one-to-one function \( i \mapsto j(i) \) from \( I \) into \( J - K \). Set \( x'_i = x_i + y_{j(i)} \) for each \( i \in I \) and define \( S^* = S + \sum_{i \in I} \{x'_i\} \).

Obviously, \( S^* \supseteq S \) and \( S^* + p^\alpha G[p] = G[p] \). Moreover, \( S^* \cap T = 0 \); for if \( 0 \neq (s + \sum n_i x'_i) \in T \) where \( s \in S \) and \( n_i \) is an integer satisfying \( 0 \leq n_i < p \), then

\[
\sum n_i y_{j(i)} \in \{R, T\} = R + \sum_{j \in K} \{y_j\},
\]

which implies that \( \sum n_i y_{j(i)} = 0 \) in view of the decomposition

\[
G[p] = R + \sum_{j \in K} \{y_j\} + \sum_{j \in J - K} \{y_j\},
\]

since \( j(i) \in J - K \). We conclude that \( n_i = 0 \) for each \( i \) and that \( 0 \neq s \in T \), but this is a contradiction that \( S \cap T = 0 \). Therefore \( S^* \cap T = 0 \), and the lemma is proved.
Theorem 1. Let $G$ be an infinite reduced $p$-group. Then either $G$ contains an $l$-pure subgroup $H$ of $G$ such that

(A) $|H| = |G|$ and $l(H) < l(G)$,

or $G$ contains a subsocle $S$ such that

(B) $|G[p] / S| = |G|$ and $\{p^n G[p], S \} = G[p]$ for each $\alpha < l(G)$.

Proof. Let $\lambda$ denote $l(G)$ and let $m = |G| = |G[p]|$. There are two cases to consider depending on whether $\lambda$ is an isolated or a limit ordinal.

Case 1. $\lambda - 1$ exists. Let $G[p] = S + p^{\lambda - 1} G[p]$. If $|p^{\lambda - 1} G[p]| = m$, then (B) holds for $S$. However, if $|p^{\lambda - 1} G[p]| < m$, then $|S| = m$. Choose $H$ maximal in $G$ with respect to $H[p] = S$. By [6, Proposition 2], $H$ is $p^\lambda$-pure in $G$. Thus $H$ is an $l$-pure subgroup of $G$ satisfying (A).

Case 2. $\lambda$ is a limit. There exists an ascending chain

$0 = S_0 \subseteq S_1 \subseteq S_2 \subseteq \cdots \subseteq S_\alpha \subseteq \cdots$, $\alpha < \lambda,$

of subsocles $S_\alpha$ of $G$ such that $S_\alpha + p^n G[p] = G[p]$. If $|S_\alpha| = m$ for any $\alpha < \lambda$, then, as in Case 1, we take $H$ maximal in $G$ with respect to $H[p] = S$, and obtain an $l$-pure subgroup $H$ of $G$ satisfying condition (A). Thus we may assume that $|S_\alpha| < m$ for each $\alpha < \lambda$. For simplicity of notation, let $m_\alpha = |S_\alpha|$. We shall employ Lemma 1 in order to obtain a subsocle $S$ of $G$ satisfying condition (B). Suppose $\mu < \lambda$ and that for each $\alpha < \mu$ a subsocle $T_\alpha$ of $G$ has been chosen such that, for suitable choices of the $S_\alpha$'s (satisfying $S_\alpha + p^n G[p] = G[p]$), we have the following:

(a) $T_\alpha \subseteq T_\beta$ if $\alpha < \beta < \mu$,

(b) $S_\alpha \cap T_\beta = 0$ if $\alpha, \beta < \mu$,

(c) $|T_\alpha| = |S_\alpha| = m_\alpha$ if $\alpha < \mu$.

We wish to define $T_\mu$ such that conditions (a)–(c) continue to hold for $\alpha, \beta < \mu$. If $\mu - 1$ exists, let $T_\mu$ be any subsocle of $G$ containing $T_{\mu - 1}$ that has cardinality $m_\mu = |S_\mu|$ and has trivial intersection with $S_\mu$; recall that $m_\mu < m$, so this is possible. By Lemma 1, we can choose $S_\mu \supseteq S_{\mu - 1}$ such that $S_\mu \cap T_\mu = 0$; we remark that $|S_\mu|$ is independent of the choice of $S_\mu$. Clearly, conditions (a)–(c) hold for $\alpha, \beta < \mu$.

If $\mu$ is a limit ordinal, set $T = \bigcup_{\alpha < \mu} T_\alpha$ and observe that

$$|T| = \sup_{\alpha < \mu} \{|T_\alpha|\} = \sup_{\alpha < \mu} \{m_\alpha\} = \sup_{\alpha < \mu} \{|S_\alpha|\} \leq |S_\mu| = m_\mu < m.$$

Therefore, we can extend $T$ to a subsocle $T_\mu$ satisfying $|T_\mu| = m_\mu$ and $(\bigcup_{\alpha < \mu} S_\alpha) \cap T_\mu = 0$. Again, by Lemma 1, we can choose $S_\mu \supseteq \bigcup_{\alpha < \mu} S_\alpha$ such that $S_\mu \cap T_\mu = 0$, and conditions (a)–(c) remain valid for $\alpha, \beta < \mu$. This completes the induction, and we may assume (a)–(c) for all $\alpha < \lambda$.

Set $S = \bigcup_{\alpha < \lambda} S_\alpha$ and $T = \bigcup_{\alpha < \lambda} T_\alpha$. Then $S \cap T = 0$, and from the equation
\[ S_\alpha + p^\alpha G[p] = G[p] \] we see that \( \{ S, p^\alpha G[p] \} = G[p] \) for each \( \alpha < \lambda \). Since
\[ |T| = \sup_{\alpha < \lambda} \{|T_\alpha|\} = \sup_{\alpha < \lambda} \{m_\alpha\} = |S|, \]
we conclude that \( |G[p]/S| = |G| \) if \( |S| = |G| \). Trivially, \( |G[p]/S| = |G| \) if \( |S| < |G| \). Thus, in either case, the subsocle \( S \) satisfies condition (B).

**Theorem 2.** Let \( G \) be a \( p \)-group. The number of \( l \)-pure subgroups of \( G \) is \( 2^{|G|} \) unless the reduced part of \( G \) is finite and the divisible part of \( G \) is zero or a single \( z(p^{\infty}) \); in the exceptional case, the number of \( l \)-pure subgroups of \( G \) is finite.

**Proof.** First, suppose that \( G \) is divisible and let \( G = \sum_m z(p^\infty) \). If \( m \) is uncountable, then \( m = |G| \). Thus if \( m \) is uncountable, obviously \( G \) has \( 2^{|G|} \) direct summands. In [4] it was shown that \( z(p^{\infty}) + z(p^{\infty}) \) has a continuum number of direct summands. It follows easily that if \( G = \sum_m z(p^\infty) \) with \( m \geq 2 \), then \( G \) has \( 2^{|G|} \) direct summands. Any summand is, of course, \( l \)-pure. Now let \( G \) be an arbitrary \( p \)-group, and write \( G = D + R \) where \( D \) is divisible and \( R \) is reduced. If \( |R| = |G| = \aleph_0 \), then we may assume that \( G = R \) because an \( l \)-pure subgroup of \( R \) is an \( l \)-pure subgroup of \( G \). Hence we may assume that \( G \) is either an infinite reduced group or else \( G = D + A \) where \( A \) is finite and \( D \) is zero or \( z(p^{\infty}) \); the other possibility is for \( G = D + R \) where \( |G| = |D| \) and \( D \) is the sum of two or more copies of \( z(p^{\infty}) \), but in this case there is no loss of generality in assuming that \( G = D \), which we have already considered. If \( G = D + A \) where \( A \) is finite and \( D \) is zero or \( z(p^{\infty}) \), then, as proved in [4], \( G \) has only a finite number of pure subgroups, so \( G \) has only a finite number of \( l \)-pure subgroups (each of which is actually a direct summand of \( G \)).

Now \( G \) is an infinite reduced group, and we want to show that \( G \) has \( 2^{|G|} \) \( l \)-pure subgroups. The proof is by induction on \( l(G) \). If \( l(G) \) is 1, then \( G \) is a vector space over \( Z/pZ \). Since \( Z/pZ \) is finite and \( G \) is infinite, \( \dim(G) = |G| \). Hence \( G \) has \( 2^{|G|} \) direct summands, so \( G \) has \( 2^{|G|} \) \( l \)-pure subgroups. We proceed to the general case. If \( G \) has an \( l \)-pure subgroup \( H \) satisfying condition (A) of Theorem 1, then the transitivity of \( l \)-purity [10] and the induction hypothesis yield \( 2^{|G|} \) \( l \)-pure subgroups of \( G \). Therefore, we may assume, by Theorem 1, that \( G \) has a subsocle \( S \) satisfying condition (B). Since \( |G[p]/S| = |G| \), there exist \( 2^{|G|} \) subsocles \( T \) of \( G \) containing \( S \). Since \( \{ p^\alpha G[p], T \} = G[p] \), if \( \alpha < l(G) \), for any subsocle \( T \) of \( G \) containing \( S \), it follows from [6, Proposition 1] that any subgroup \( H \) of \( G \) maximal in \( G \) with respect to \( H[p] = T \) is \( l \)-pure in \( G \). Thus there exist \( 2^{|G|} \) \( l \)-pure subgroups of \( G \).

**Corollary 1.** The number of \( l \)-pure subgroups and the number of isotype subgroups coincide for any \( p \)-group \( G \). Therefore, the number of isotype subgroups of a \( p \)-group \( G \) is given by Theorem 2.
Proof. Since any $l$-pure subgroup is necessarily an isotype subgroup by [8, Theorem 15] and since $G$ has at most $2^{|G|}$ subgroups, the corollary follows immediately for all $G$ not in the exceptional case of Theorem 2. Thus we need only to prove that $G$ has the same number of $l$-pure subgroups as it does isotype subgroups in case $G = D + A$ where $A$ is finite and $D$ is zero or $Z(p^\infty)$. However, it is well known (and easy to prove) that any pure subgroup of such a group $G$ is necessarily a direct summand of $G$, so the concepts of pure subgroup, isotype subgroup, $l$-pure subgroup, and direct summand all coincide in this case.

We now present the result that we promised in the introduction concerning the number of nonisomorphic isotype subgroups.

Theorem 3. Let $G$ be a direct sum of reduced countable $p$-groups. If $G$ is unbounded, has countable length, and if $G$ has cardinality $\aleph_\alpha$ for some countable ordinal $\alpha$, then $G$ has exactly a continuum number of nonisomorphic isotype subgroups.

Proof. In order to show that $G$ has at least a continuum number of isomorphically distinct isotype subgroups, we observe that $G = H + K$ where $H$ is an unbounded direct sum of cyclic groups. Indeed, any unbounded reduced countable $p$-group has such a decomposition in view of Ulm's theorem. It obviously suffices to show that $H$ has a continuum number of isomorphically distinct isotype subgroups, but clearly $H$ even has a continuum number of isomorphically distinct direct summands.

Now we prove that $G$ has at most a continuum number of isomorphically distinct isotype subgroups. In order to prove this, we need the author's theorem that implies that any isotype subgroup $H$ of $G$ is itself a direct sum of countable groups [6], and therefore $H$ is determined, up to isomorphism, by its Ulm invariants [9], [5]. It remains only to observe that since $l(G)$ is countable $H$ has at most a countable number of nonzero Ulm invariants each of which is a cardinal not exceeding $|G| = \aleph_\alpha < \aleph_\omega$.

Remarks. One interesting consequence of Theorem 3 is the following. Suppose that the unbounded reduced $p$-group $G$ is restricted to countable length and is a direct sum of countable groups. Then within the bounds $\aleph_0 \leq |G| < \aleph_\omega$ an increase in the size of $G$ does not increase the number of nonisomorphic isotype subgroups of $G$. In particular, if the (generalized) continuum hypothesis is assumed, $G$ can be quite large and have relatively few isomorphically distinct isotype subgroups. Finally, we remark that if it is a simple exercise to show that if $G$ is a bounded $p$-group, then $G$ has exactly $\prod_{i=0}^{k} m_i$ isomorphically distinct isotype subgroups where $p^{k+1}$ is the smallest bound on $G$ and $m_i$ is the number of cardinals not exceeding the $i$th Ulm invariant of $G$. 

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