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THE NUMBER OF ISOTYPE AND l-PURE SUBGROUPS OF AN ABELIAN p-GROUP

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ABSTRACT. The number of isotype subgroups of an abelian p-group G is determined. This solves a recent problem of Fuchs. Actually, we accomplish slightly more. Define a subgroup H of an abelian p-group G to be an l-pure subgroup of G if, for some ordinal λ, H is p^λ-pure in G and $p^\lambda H$ is divisible. We compute the number of l-pure subgroups of G and show that the number of l-pure subgroups and the number of isotype subgroups of G coincide. Our final result deals with the number of nonisomorphic isotype subgroups of G when G is a direct sum of countable groups.

In this paper all groups are abelian, and "abelian group" is shortened to "group". For the most part, our notation and terminology agree with **[2]**. However, for the convenience of the reader we list the meaning of the following frequently used symbols and terms.

$+$: direct sum.

$\sum$: direct sum (of an infinite family).

$|X|$: cardinality of X.

$l(G)$: the length of a reduced p-group G.

$p^\alpha G[p]$: $(p^\alpha G)[p]$.

c: cardinality of the continuum, $2^{\aleph_0}$.

ω, Ω: the first infinite ordinal and the first uncountable ordinal, respectively.

isotype subgroup: H is an isotype subgroup of the p-group G if $p^\alpha G \cap H = p^\alpha H$ for every ordinal α.

l-pure subgroup: H is an l-pure subgroup of the p-group G if, for some ordinal λ, H is p^λ-pure in G and $p^\lambda H$ is divisible; recall that H is p^λ-pure in G if $H \rightarrowtail G \twoheadrightarrow G/H$ belongs to p^λ $\mathrm{Ext}(G/H, H)$.

subsocle: a subsocle of a p-group G is any subgroup of $G[p]$.

In **[2]** Fuchs raised the question: What is the cardinality of the set of all pure subgroups of a group G? The answer to this problem was provided by the combination of two papers, one by Boyer **[1]** and the other by Hill **[4]**. In his new book **[3]**, Fuchs has presented as Problem 20(a):

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What is the cardinality of the set of [nonisomorphic] isotype subgroups of a p-group G? We can answer completely the primary question: What is the cardinality of the set of isotype subgroups of G? We suggest that the theory of abelian groups may not yet be advanced enough to solve, in a meaningful way, the secondary problem of determining for all p-groups G the number of nonisomorphic isotype subgroups. However, this is certainly possible for many special cases of G. For example, if G is a p-group of cardinality c without elements of infinite height having a countable basic subgroup, then G has 2^c isomorphically distinct isotype subgroups according to Theorem 2.3 of [7]. In the present paper (Theorem 3), we determine the number of isomorphically distinct isotype subgroups of a p-group G of countable length that can be written as a direct sum of less than $\aleph_\Omega$ countable groups.

LEMMA 1. *Let G be a p-group. Suppose, for an ordinal α, that $m_\alpha < m \geqq \aleph_0$ where $m_\alpha = |G[p]/p^\alpha G[p]|$ and $m = |G[p]|$. Let S be a subsocle of G such that $S \cap p^\alpha G[p] = 0$. If T is a subsocle of G such that $|T| < m$ and such that $T \cap S = 0$, then there exists $S^* \subseteq G[p]$ such that*:

(1) $S^* \supseteq S$,

(2) $S^* + p^\alpha G[p] = G[p]$,

(3) $S^* \cap T = 0$.

PROOF. First, choose $R \supseteq S$ such that R is maximal in $G[p]$ with respect to $R \cap p^\alpha G[p] = 0$. Then $S^* = R$ satisfies conditions (1) and (2). If $R \cap T = 0$, the proof is finished. Thus we may assume that $R \cap T \neq 0$. Write $p^\alpha G[p] = \sum_{j \in J} \{y_j\}$, and note that $|J| = m$. Since $G[p] = R + \sum_{j \in J} \{y_j\}$, we have, for a suitable choice of the basis $\{y_j\}$,that $\{R, T\} = R + \sum_{j \in K} \{y_j\}$ for some subset K of J. Also, $|K| < |J| = m$ since T has cardinality less than m. Write $R = S + W$, and set $W = \sum_{i \in I} \{x_i\}$. Observe that $|J - K| = m > m_\alpha \geqq |I|$, so there exists a one-to-one function $i \to j(i)$ from I into $J - K$. Set $x_i' = x_i + y_{j(i)}$ for each $i \in I$ and define $S^* = S + \sum_{i \in I} \{x_i'\}$.

Obviously, $S^* \supseteq S$ and $S^* + p^\alpha G[p] = G[p]$. Moreover, $S^* \cap T = 0$; for if $0 \neq (s + \sum n_i x_i') \in T$ where $s \in S$ and n_i is an integer satisfying $0 \leqq n_i < p$, then

$$\sum n_i y_{j(i)} \in \{R, T\} = R + \sum_{j \in K} \{y_j\},$$

which implies that $\sum n_i y_{j(i)} = 0$ in view of the decomposition

$$G[p] = R + \sum_{j \in K} \{y_j\} + \sum_{j \in J-K} \{y_j\}$$

since $j(i) \in J - K$. We conclude that $n_i = 0$ for each i and that $0 \neq s \in T$, but this is a contradiction that $S \cap T = 0$. Therefore $S^* \cap T = 0$, and the lemma is proved.

THEOREM 1. *Let G be an infinite reduced p-group. Then either G contains an l-pure subgroup H of G such that*

(A) $|H|=|G|$ and $l(H)<l(G)$,

or G contains a subsocle S such that

(B) $|G[p]/S|=|G|$ *and* $\{p^\alpha G[p], S\}=G[p]$ *for each* $\alpha<l(G)$.

PROOF. Let λ denote $l(G)$ and let $m=|G|=|G[p]|$. There are two cases to consider depending on whether λ is an isolated or a limit ordinal.

Case 1. $\lambda-1$ exists. Let $G[p]=S+p^{\lambda-1}G[p]$. If $|p^{\lambda-1}G[p]|=m$, then (B) holds for S. However, if $|p^{\lambda-1}G[p]|<m$, then $|S|=m$. Choose H maximal in G with respect to $H[p]=S$. By [**6**, Proposition 2], H is p^λ-pure in G. Thus H is an l-pure subgroup of G satisfying (A).

Case 2. λ is a limit. There exists an ascending chain

$$0 = S_0 \subseteq S_1 \subseteq S_2 \subseteq \cdots \subseteq S_\alpha \subseteq \cdots, \qquad \alpha < \lambda,$$

of subsocles S_α of G such that $S_\alpha+p^\alpha G[p]=G[p]$. If $|S_\alpha|=m$ for any $\alpha<\lambda$, then, as in Case 1, we take H maximal in G with respect to $H[p]=S_\alpha$ and obtain an l-pure subgroup H of G satisfying condition (A). Thus we may assume that $|S_\alpha|<m$ for each $\alpha<\lambda$. For simplicity of notation, let $m_\alpha=|S_\alpha|$. We shall employ Lemma 1 in order to obtain a subsocle S of G satisfying condition (B). Suppose $\mu<\lambda$ and that for each $\alpha<\mu$ a subsocle T_α of G has been chosen such that, for suitable choices of the S_α's (satisfying $S_\alpha+p^\alpha G[p]=G[p]$), we have the following:

(a) $T_\alpha\subseteq T_\beta$ if $\alpha<\beta<\mu$,

(b) $S_\alpha\cap T_\beta=0$ if $\alpha, \beta<\mu$,

(c) $|T_\alpha|=|S_\alpha|=m_\alpha$ if $\alpha<\mu$.

We wish to define T_μ such that conditions (a)–(c) continue to hold for $\alpha, \beta\leqq\mu$. If $\mu-1$ exists, let T_μ be any subsocle of G containing $T_{\mu-1}$ that has cardinality $m_\mu=|S_\mu|$ and has trivial intersection with S_μ; recall that $m_\mu<m$, so this is possible. By Lemma 1, we can choose $S_\mu\supseteq S_{\mu-1}$ such that $S_\mu\cap T_\mu=0$; we remark that $|S_\alpha|$ is independent of the choice of S_α. Clearly, conditions (a)–(c) hold for $\alpha, \beta\leqq\mu$.

If μ is a limit ordinal, set $T=\bigcup_{\alpha<\mu} T_\alpha$ and observe that

$$|T| = \sup_{\alpha<\mu}\{|T_\alpha|\} = \sup_{\alpha<\mu}\{m_\alpha\} = \sup_{\alpha<\mu}\{|S_\alpha|\} \leqq |S_\mu| = m_\mu < m.$$

Therefore, we can extend T to a subsocle T_μ satisfying $|T_\mu|=m_\mu$ and $(\bigcup_{\alpha<\mu} S_\alpha)\cap T_\mu=0$. Again, by Lemma 1, we can choose $S_\mu\supseteq\bigcup_{\alpha<\mu} S_\alpha$ such that $S_\mu\cap T_\mu=0$, and conditions (a)–(c) remain valid for $\alpha, \beta<\mu$. This completes the induction, and we may assume (a)–(c) for all $\alpha<\lambda$. Set $S=\bigcup_{\alpha<\lambda} S_\alpha$ and $T=\bigcup_{\alpha<\lambda} T_\alpha$. Then $S\cap T=0$, and from the equation

$S_\alpha + p^\alpha G[p] = G[p]$ we see that $\{S, p^\alpha G[p]\} = G[p]$ for each $\alpha < \lambda$. Since

$$|T| = \sup_{\alpha<\lambda} \{|T_\alpha|\} = \sup_{\alpha<\lambda} \{m_\alpha\} = |S|,$$

we conclude that $|G[p]/S| = |G|$ if $|S| = |G|$. Trivially, $|G[p]/S| = |G|$ if $|S| < |G|$. Thus, in either case, the subsocle S satisfies condition (B).

THEOREM 2. *Let G be a p-group. The number of l-pure subgroups of G is $2^{|G|}$ unless the reduced part of G is finite and the divisible part of G is zero or a single $z(p^\infty)$; in the exceptional case, the number of l-pure subgroups of G is finite.*

PROOF. First, suppose that G is divisible and let $G = \sum_m z(p^\infty)$. If m is uncountable, then $m = |G|$. Thus if m is uncountable, obviously G has $2^{|G|}$ direct summands. In **[4]** it was shown that $z(p^\infty) + z(p^\infty)$ has a continuum number of direct summands. It follows easily that if $G = \sum_m z(p^\infty)$ with $m \geqq 2$, then G has $2^{|G|}$ direct summands. Any summand is, of course, l-pure. Now let G be an arbitrary p-group, and write $G = D + R$ where D is divisible and R is reduced. If $|R| = |G| \geqq \aleph_0$, then we may assume that $G = R$ because an l-pure subgroup of R is an l-pure subgroup of G. Hence we may assume that G is either an infinite reduced group or else $G = D + A$ where A is finite and D is zero or $z(p^\infty)$; the other possibility is for $G = D + R$ where $|G| = |D|$ and D is the sum of two or more copies of $z(p^\infty)$, but in this case there is no loss of generality in assuming that $G = D$, which we have already considered. If $G = D + A$ where A is finite and D is zero or $z(p^\infty)$, then, as proved in **[4]**, G has only a finite number of pure subgroups, so G has only a finite number of l-pure subgroups (each of which is actually a direct summand of G).

Now G is an infinite reduced group, and we want to show that G has $2^{|G|}$ l-pure subgroups. The proof is by induction on $l(G)$. If $l(G)$ is 1, then G is a vector space over Z/pZ. Since Z/pZ is finite and G is infinite, $\dim(G) = |G|$. Hence G has $2^{|G|}$ direct summands, so G has $2^{|G|}$ l-pure subgroups. We proceed to the general case. If G has an l-pure subgroup H satisfying condition (A) of Theorem 1, then the transitivity of l-purity **[10]** and the induction hypothesis yield $2^{|G|}$ l-pure subgroups of G. Therefore, we may assume, by Theorem 1, that G has a subsocle S satisfying condition (B). Since $|G[p]/S| = |G|$, there exist $2^{|G|}$ subsocles T of G containing S. Since $\{p^\alpha G[p], T\} = G[p]$, if $\alpha < l(G)$, for any subsocle T of G containing S, it follows from **[6**, Proposition 1] that any subgroup H of G maximal in G with respect to $H[p] = T$ is l-pure in G. Thus there exist $2^{|G|}$ l-pure subgroups of G.

COROLLARY 1. *The number of l-pure subgroups and the number of isotype subgroups coincide for any p-group G. Therefore, the number of isotype subgroups of a p-group G is given by Theorem 2.*

PROOF. Since any l-pure subgroup is necessarily an isotype subgroup by [**8**, Theorem 15] and since G has at most $2^{|G|}$ subgroups, the corollary follows immediately for all G not in the exceptional case of Theorem 2. Thus we need only to prove that G has the same number of l-pure subgroups as it does isotype subgroups in case $G=D+A$ where A is finite and D is zero or $z(p^\infty)$. However, it is well known (and easy to prove) that any pure subgroup of such a group G is necessarily a direct summand of G, so the concepts of pure subgroup, isotype subgroup, l-pure subgroup, and direct summand all coincide in this case.

We now present the result that we promised in the introduction concerning the number of nonisomorphic isotype subgroups.

THEOREM 3. *Let G be a direct sum of reduced countable p-groups. If G is unbounded, has countable length, and if G has cardinality $\aleph_\alpha$ for some countable ordinal α, then G has exactly a continuum number of nonisomorphic isotype subgroups.*

PROOF. In order to show that G has at least a continuum number of isomorphically distinct isotype subgroups, we observe that $G=H+K$ where H is an unbounded direct sum of cyclic groups. Indeed, any unbounded reduced countable p-group has such a decomposition in view of Ulm's theorem. It obviously suffices to show that H has a continuum number of isomorphically distinct isotype subgroups, but clearly H even has a continuum number of isomorphically distinct direct summands.

Now we prove that G has at most a continuum number of isomorphically distinct isotype subgroups. In order to prove this, we need the author's theorem that implies that any isotype subgroup H of G is itself a direct sum of countable groups [**6**], and therefore H is determined, up to isomorphism, by its Ulm invariants [**9**], [**5**]. It remains only to observe that since $l(G)$ is countable H has at most a countable number of nonzero Ulm invariants each of which is a cardinal not exceeding $|G|=\aleph_\alpha<\aleph_\Omega$.

REMARKS. One interesting consequence of Theorem 3 is the following. Suppose that the unbounded reduced p-group G is restricted to countable length and is a direct sum of countable groups. Then within the bounds $\aleph_0\leqq|G|<\aleph_\Omega$ an increase in the size of G does not increase the *number* of nonisomorphic isotype subgroups of G. In particular, if the (generalized) continuum hypothesis is assumed, G can be quite large and have relatively few isomorphically distinct isotype subgroups. Finally, we remark that it is a simple exercise to show that if G is a bounded p-group, then G has exactly $\prod_{i=0}^{k} m_i$ isomorphically distinct isotype subgroups where p^{k+1} is the smallest bound on G and m_i is the number of cardinals not exceeding the ith Ulm invariant of G.

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