ON FUNCTIONS OF BOUNDED ROTATION

J. W. NOONAN

Abstract. For fixed \( k \geq 2 \), denote by \( V_k \) and \( R_k \) the classes of functions regular in the unit disc and having boundary and radial rotation, respectively, at most \( k \pi \). The concept of order of a function is defined for both \( V_k \) and \( R_k \). For functions in these classes, the growth of integral and coefficient means is studied in terms of the order of the function. Some length-area results are also obtained.

1. Introduction. For fixed \( k \geq 2 \), denote by \( V_k \) the class of normalized functions, analytic in the unit disc \( \gamma \), which have boundary rotation at most \( k \pi \). That is, a function \( f \) of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

belongs to \( V_k \) if and only if \( f'(z) \neq 0 \) for \( z \in \gamma \) and, with \( z = re^{i\theta} \),

\[
\int_0^{2\pi} \left| \frac{zf'(z)}{f'(z)} \right| d\theta \leq k \pi.
\]

Equivalently [7], \( f \in V_k \) if and only if there exists a real-valued function \( \mu \) on \( [0, 2\pi] \) with \( \int_0^{2\pi} d\mu(t) = 2 \) and \( \int_0^{2\pi} |d\mu(t)| \leq k \) such that

\[
f'(z) = \exp\left(-\int_0^{2\pi} \log(1 - ze^{-it}) \, d\mu(t)\right).
\]

Note that \( V_2 \) is the class of convex functions.

With \( k \geq 2 \) still fixed, denote by \( R_k \) the class of functions \( g \) of the form

\[
g(z) = z + \sum_{n=2}^{\infty} b_n z^n
\]

which have radial rotation at most \( k \pi \). That is, \( g \in R_k \) if and only if \( g(z)/z \neq 0 \)

1 The author is a N.R.C.-N.R.L. Postdoctoral Resident Research Associate.
for \( z \in \gamma \) and, with \( z = re^{i\theta} \),

\[
(1.5) \quad \oint_{\gamma} \frac{\text{Re} \frac{zg'(z)}{g(z)}}{d\theta} \leq k\pi.
\]

One can easily show that \( g \in R_k \) if and only if there exists a real-valued function \( m \) on \([0, 2\pi]\) with \( \oint_{\gamma} dm(t) = 2 \) and \( \oint_{\gamma} |dm(t)| \leq k \) such that

\[
(1.6) \quad g(z) = z \exp \left\{ -\oint_{\gamma} \log(1 - ze^{-it}) \, dm(t) \right\}.
\]

Note that \( R_2 \) is the class of starlike functions, and that \( f \in V_k \) if and only if \( g \in R_k \), where \( g(z) = zf'(z) \).

We now proceed to define the concept of order for functions in \( V_k \) and \( R_k \). Let \( f \in V_k \) be given by (1.3). We shall require that \( \mu \) be normalized by the conditions \( \int_{\gamma} \mu(t) \, dt = 0 \) and \( \mu(t) = (\mu(t+0) + \mu(t-0))/2 \), in which case (1.3) defines a unique relationship between \( f \) and \( \mu \). Writing \( \mu = \mu^+ - \mu^- \) for the canonical decomposition of \( \mu \) into the difference of nondecreasing functions, we define

\[
(1.3) \quad \alpha(f) = \max \{ \mu^+(t + 0) - \mu^+(t - 0) : t \in [0, 2\pi] \}
\]

to be the order of \( f \). Since \( \mu \) is of bounded variation, \( \alpha(f) \) exists and is the largest nonnegative jump of \( \mu \). Also note that if \( \alpha(f) = \mu^+(\theta + 0) - \mu^+(\theta - 0) > 0 \), then the normalization condition on \( \mu \) implies that \( \mu^- \) is continuous at \( \theta \).

If \( g \in R_k \) is given by (1.6), we shall require \( m \) to be normalized as above and write \( m = m^+ - m^- \) for the canonical decomposition of \( m \). Now we define

\[
(1.6) \quad \beta(g) = \max \{ m^+(t + 0) - m^+(t - 0) : t \in [0, 2\pi] \}
\]

to be the order of \( g \). Again note that if \( \beta(g) = m^+(\theta + 0) - m^+(\theta - 0) > 0 \), then \( m^- \) is continuous at \( \theta \). For \( k = 2 \), one has Pommerenke's definition of the order of a starlike function [9].

In [3], F. Holland and D. K. Thomas obtained results concerning the order of a starlike function. Since every convex function is starlike, it is natural to ask whether analogues of the results in [3] are true for \( V_2 \), or more generally, for \( V_k \) and \( R_k \). The purpose of this paper is to extend the results in [3] to \( V_k \) and \( R_k \). It is interesting to note that to derive these results for \( V_k \) (and even for \( V_2 \)), it seems necessary to consider the corresponding class \( R_k \). The reason is that the representation (1.3) is for \( f' \), and many of the problems considered here seem to require a representation for \( f \).
2. Preliminary theorems. If $F$ is analytic in $y$, set

$$M(r, F) = \max_{|z|=r} |F(z)|.$$  

Also, with $z=re^{i\theta}$, set

$$L(r, F) = \int_{0}^{2\pi} |zF'(z)| \, d\theta$$

and

$$A(r, F) = \frac{1}{\pi} \int_{0}^{2\pi} \int_{0}^{r} |F'(re^{i\theta})|^2 \, \rho \, d\rho \, d\theta.$$  

Then $L(r, F)$ is the length of the image of $\{z:|z|=r\}$ under $F$, and $\pi A(r, F)$ is the area of the image of $\{z:|z|\leq r\}$ under $F$. Throughout this paper we shall use the notation $f$ and $\alpha = \alpha(f)$ for a $V_k$ function and its order, and $g$ and $\beta = \beta(g)$ for an $R_k$ function and its order.

**Theorem 2.1.** (i) If $g \in R_k$, then

$$\frac{\log M(r, g)}{r - \log(1 - r)} = \alpha,$$

(ii) If $f \in V_k$, then

$$\frac{\log M(r, f')}{r - \log(1 - r)} = \alpha,$$

$$\frac{\log M(r, f)}{r - \log(1 - r)} = \max\{0, \alpha - 1\}.$$  

The proofs of (i) and the first part of (ii) are essentially the same as the proof of [8, Theorem 1]. The second statement in (ii) is proved in [6].

**Theorem 2.2.** If $g \in R_k$ and $\beta > 0$, then

$$\lim_{r \to 1} \frac{(1 - r)M'(r, g)}{M(r, g)} = \beta,$$

where $M'(r, g)$ is the left derivative.

Since the proof is essentially the same as that of [9, Theorem 1], we again omit the details. There is, however, one point which should be mentioned. Suppose $\beta = m^+(\theta + 0) - m^+(\theta - 0)$. Since $\beta > 0$, it follows that $m^-$ is continuous at $\theta$. This allows us to use the bounded convergence theorem in the same manner as Pommerenke, and the proof is then easily completed.

**Theorem 2.3.** If $g \in R_k$ is given by (1.4), then

(i) $n |b_n| r^n \leq 3kM(r, g)$,

(ii) $L(r, g) = O(1)M(r, g)$ if $\beta > 0$.

The constant $O(1)$ depends on $\beta$, and hence on $g$.  

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PROOF. Since $g \in R_k$, $f \in V_k$ where $zf'(z) = g(z)$. Thus (i) follows immediately from [10, Theorem 1]. In the case $k = 2$, Pommerenke [9] proved (ii) by using Theorem 2.2 above for $k = 2$. His proof generalizes easily for $k > 2$.

In our next theorem we derive a relationship between the classes $V_k$ and $R_k$. This theorem will enable us to extend our results for $R_k$ to the class $V_k$. We note first that any $f \in V_k$ is finitely valent [2], and in particular $f$ has only finitely many zeros in $\gamma$.

**Theorem 2.4.** If $f \in V_k$, denote by $\{\delta_j\}_{j=0}^p$ the zeros of $f$ in $\gamma$, where we choose $\delta_0 = 0$. Write $G(z) = c f(z) h(z)$ where $c = \prod_{j=1}^p (z - \delta_j)$ and $h$ is given by $h(z) = \prod_{j=1}^p (z - \delta_j)$. Then there exists a constant $K$ such that $G \in R_k$. Also, $\beta(G) = \max\{0, \alpha(f) - 1\}$.

**Remark.** If $f$ vanishes only at the origin (i.e. $p = 0$), then by convention, $\prod_{j=1}^p = 1$.

**Proof.** Since $f'(z) \neq 0$ for $z \in \gamma$, all zeros of $f$ are simple, and so $G$ vanishes only for $z = 0$. Thus, from the definition of $G$, with $z = re^{i\theta}$,
\[
\int_0^{2\pi} |d \arg G(z)| \, d\theta \leq \int_0^{2\pi} |d \arg f(z)| \, d\theta + 2\pi p
\]
for $r > \max_j \{|\delta_j|\}$. Since neither $f$ nor $f'$ vanish for $r > \max_j \{|\delta_j|\}$, a result of Biernacki [1] gives
\[
\int_0^{2\pi} |d \arg f(z)| \, d\theta \leq \int_0^{2\pi} \left| \frac{(zf'(z))'}{f'(z)} \right| \, d\theta \leq k\pi,
\]
and so
\[
\int_0^{2\pi} \left| \frac{zG'(z)}{G(z)} \right| \, d\theta = \int_0^{2\pi} |d \arg G(z)| \, d\theta \leq (k + 2p)\pi
\]
for $r > \max_j \{|\delta_j|\}$. The first part of the theorem is now obvious. It is also clear from the definition of $G$ that
\[
\lim_{r \to 1} \frac{\log M(r, G)}{-\log(1 - r)} = \lim_{r \to 1} \frac{\log M(r, f)}{-\log(1 - r)},
\]
and so from Theorem 2.1 we have $\beta(G) = \max\{0, \alpha(f) - 1\}$.

3. **Integral and coefficient means.** In this section, if $\lambda > 0$ and $F$ is analytic in $\gamma$ we set, for $0 \leq r < 1$,
\[
I_\lambda(r, F) = \frac{1}{2\pi} \int_0^{2\pi} |F(re^{i\theta})|^\lambda \, d\theta.
\]
With this notation we have

**Theorem 3.1.** If \( g \in R_k \) and \( \lambda > 0 \), then

\[
\lim_{r \to 1} \frac{\log I_\lambda(r, g)}{-\log(1 - r)} = \max\{0, \beta \lambda - 1\}.
\]

**Proof.** From [4, p. 214] we have

\[
I_\lambda(r, g) \geq \frac{\lambda}{\pi} \int_0^r \frac{M(\rho, g)^\lambda}{\rho} \, d\rho.
\]

It is clear from Theorem 2.1(i) that for any \( \varepsilon > 0 \), \( M(r, g) \geq B(\varepsilon)(1 - r)^{-\beta + \varepsilon} \)

and so from (3.1)

\[
\lim \inf_{r \to 1} \frac{\log I_\lambda(r, g)}{-\log(1 - r)} \geq \max\{0, \beta \lambda - 1\}.
\]

Next, as in [3], we note that with \( z = re^{i\theta} \),

\[
r(d/dr) \log |g(z)| = \Re (zg'(z)/g(z)),
\]

and so

\[
r \frac{d}{dr} I_\lambda(r, g) = \frac{\lambda}{2\pi} \int_0^{2\pi} |g(z)|^4 \Re \frac{zg'(z)}{g(z)} \, d\theta \leq \frac{\lambda \kappa}{2} M(r, g)^\lambda,
\]

where we have used (1.5). Hence

\[
I_\lambda(r, g) \leq \frac{\lambda \kappa}{2} \int_0^r \frac{M(\rho, g)^\lambda}{\rho} \, d\rho,
\]

and so

\[
\lim \sup_{r \to 1} \frac{\log I_\lambda(r, g)}{-\log(1 - r)} \leq \max\{0, \beta \lambda - 1\}
\]

on using Theorem 2.1(i). Combining (3.2) and (3.3), we have Theorem 3.1.

**Theorem 3.2.** If \( f \in V_k \) and \( \lambda > 0 \), then

(i) \[
\lim_{r \to 1} \frac{\log I_\lambda(r, f^\prime)}{-\log(1 - r)} = \max\{0, \alpha \lambda - 1\}
\]

and

(ii) \[
\lim_{r \to 1} \frac{\log I_\lambda(r, f)}{-\log(1 - r)} = \max\{0, \lambda \alpha - \lambda - 1\}.
\]

**Proof.** If \( f \in V_k \) and \( g \) is defined by \( g(z) = zf'(z) \), then \( g \in R_k \) and \( \beta(g) = \alpha(f) \), and so (i) follows immediately from Theorem 3.1. In order to prove (ii), let \( G \in R_k \) be related to \( f \) as in Theorem 2.4, and denote the order of \( G \)
by $\beta = \max\{0, \alpha - 1\}$. Then with the notation of Theorem 2.4, we have
\[
I_\lambda(r, f) \leq \prod_{j=1}^{p} (1 + r/|\delta_j|)^4 I_\lambda(r, G)
\]
and, with $r > \max_j \{|\delta_j|\}$,
\[
I_\lambda(r, f) \geq \prod_{j=1}^{p} (r/|\delta_j| - 1)^4 I_\lambda(r, G),
\]
from which the result follows upon using Theorem 3.1.

We now study the growth of the coefficients of $V_k$ and $R_k$ functions. If $F$ given by $F(z) = \sum_{n=1}^{\infty} d_n z^n$ is analytic in $\gamma$, then as in [3] we define for $\lambda > 0$ and $0 \leq r < 1$,
\[
P_\lambda(r, F) = \sum_{n=1}^{\infty} n^{\lambda - 1} |d_n|^\lambda r^n.
\]
Note that $P_2(r^2, F) = A(r, F)$. We now have

**Theorem 3.3.** (i) If $g \in R_k$ and $\lambda > 0$, then
\[
\lim_{r \to 1} \frac{\log P_\lambda(r, g)}{-\log(1 - r)} = \beta \lambda.
\]
(ii) If $f \in V_k$ and $\lambda > 0$, then
\[
\lim_{r \to 1} \frac{\log P_\lambda(r, f)}{-\log(1 - r)} = \lambda \max\{0, \alpha - 1\}.
\]

**Proof.** The proof of (i) is a direct analogue of that given in [3]. However, we include the proof, since we shall need the method to prove (ii). By Theorem 2.3(i),
\[
P_\lambda(r^{k+1}, g) = \sum_{n=1}^{\infty} (n |b_n| r^n)^{\lambda} \frac{r^n}{n}
\]
\[
\leq (3kM(r, g))^\lambda \log \frac{1}{1 - r},
\]
and so from Theorem 2.1(i),
\[
(3.4) \quad \limsup_{r \to 1} \frac{\log P_\lambda(r, g)}{-\log(1 - r)} \leq \beta \lambda.
\]
Now suppose $\lambda \geq 1$. By Hölder's inequality,
\[
P_\lambda(r, g) \left( \log \frac{1}{1 - r} \right)^{\lambda - 1} \geq M(r, g)^\lambda,
\]
and so Theorem 2.1(i) gives
\[
\liminf_{r \to 1} \frac{\log P_\lambda(r, g)}{-\log(1 - r)} \geq \beta \lambda,
\]
which proves (i) for $\lambda \geq 1$. 
If $0<\lambda<1$, then again using Hölder's inequality,

$$P_\lambda(r, g) = \sum_{n=1}^{\infty} |b_n|^\lambda r^n \leq \left( \sum_{n=1}^{\infty} |b_n|^{\lambda \frac{1}{\lambda}} r^n \right)^{1-\frac{1}{\lambda}} \left( \sum_{n=1}^{\infty} |b_n|^{1+\lambda - \lambda} r^n \right)^{\lambda-1},$$

and so

$$\frac{\log P_\lambda(r, g)}{-\log(1-r)} \leq \lambda \frac{\log P_\lambda(r, g)}{-\log(1-r)} + (1 - \lambda) \frac{\log P_{1+\lambda}(r, g)}{-\log(1-r)},$$

which gives

$$\beta \leq \lambda \liminf_{r \to 1} \frac{\log P_\lambda(r, g)}{-\log(1-r)} + (1 - \lambda)(1 + \lambda)\beta$$
on using (i) for the case $\lambda \geq 1$. Hence

$$\lambda \leq \liminf_{r \to 1} \frac{\log P_\lambda(r, g)}{-\log(1-r)},$$

and (i) follows for $0<\lambda<1$ upon combining (3.4) and (3.5).

In order to prove (ii), we first note that if $f \in V_k$ is given by (1.1), then

$$n |a_n| r^n \leq (1/2\pi) L(r, f) \leq (k/2) M(r, f),$$

and so with the notation of Theorem 2.4,

$$n |a_n| r^n \leq B M(r, G),$$

where $G \in R_K$ and $2B = k \max_{|z| \leq 1} |h(z)/c|$. As in the proof of (i), we then find

$$P_{1+\lambda}(r, f) \leq B^{1+\lambda} M(r, G)^{\lambda} \log \frac{1}{1-r},$$

which gives

$$\limsup_{r \to 1} \frac{\log P_{1+\lambda}(r, f)}{-\log(1-r)} \leq \lambda \max\{0, \alpha - 1\}.$$In order to prove

$$\liminf_{r \to 1} \frac{\log P_{1+\lambda}(r, f)}{-\log(1-r)} \geq \lambda \max\{0, \alpha - 1\},$$

we proceed exactly as in the proof of (i), except that we use Theorem 2.1(ii) in place of Theorem 2.1(i). This proves Theorem 3.3.

To conclude this section, we have

**Theorem 3.4.** (i) If $g \in R_k$ is given by (1.4), then

$$\limsup_{n \to \infty} \frac{\log n |b_n|}{\log n} = \beta.$$
(ii) If \(f \in V_k\) is given by (1.1), then
\[
\limsup_{n \to \infty} \frac{\log^+ n |a_n|}{\log n} = \max\{0, x - 1\}.
\]

The proof of this theorem is essentially the same as that given in [3, Theorem 5], so we omit the details. Note that (ii) follows immediately from (i) since \(f \in V_k\) implies \(g \in R_k\) where \(g(z) = zf'(z)\).

4. Some length-area results. In this section we estimate \(L(r, F)\) in terms of \(A(r, F)^{1/2}\) for \(R_k\) and \(V_k\). We must first prove a technical lemma.

**Lemma 1.**

(i) If \(g \in R_k\), then for \(0 \leq r < 1\),
\[
rM(r, g) \leq 2^{k+1}M(r^2, g).
\]

(ii) If \(f \in V_k\) and \(G \in R_K\) is as in Theorem 2.4, then as \(r \to 1\), \(A(r, G) = O(1)A(r, f)\).

**Proof.**

If \(g \in R_k\), then \(f \in V_k\) where \(zf'(z) = g(z)\). It then follows immediately from [5, Corollary 3.2] that
\[
\left(1 - r^2\right)^{k+1} M(r, g) \leq \frac{1 + r}{1 - r} \frac{M(r, g)}{r}
\]
is a decreasing function of \(r\), which in turn gives (i).

With the notation of Theorem 2.4, \(G(z) = cf(z)/h(z)\). If \(p = 0\) in Theorem 2.4, then \(c = 1\) and \(h = 1\). Write \(z = re^{i\theta}\), and choose \(r_0 > \max\{1, |\delta_j|\}\). Then with \(r > r_0\), \(|1/h(z)|\) and \(|h'(z)/h^2(z)|\) are both bounded above, say by \(B\). Hence
\[
\left\{\int_0^{2\pi} |G'(z)|^2 d\theta\right\}^{1/2} \leq B\left\{\int_0^{2\pi} |f'(z)|^2 d\theta\right\}^{1/2} + B\left\{\int_0^{2\pi} |f(z)|^2 d\theta\right\}^{1/2}
\]
\[
\leq 2B\left\{\int_0^{2\pi} |f'(z)|^2 d\theta\right\}^{1/2},
\]
and so
\[
(4.1) \quad \int_{r_0}^r \int_0^{2\pi} |G'(pe^{i\theta})|^2 \rho \, d\rho \, d\theta \leq 4B^2 \int_{r_0}^r \int_0^{2\pi} |f'(pe^{i\theta})|^2 \rho \, d\rho \, d\theta.
\]

Straightforward computation shows \(\lim_{r \to 0} A(r, G)/A(r, f)\) exists and is finite, so there exists \(B(r_0)\) such that \(A(r, G) \leq B(r_0)A(r, f)\) for \(0 \leq r \leq r_0\). On combining this with (4.1), (ii) follows easily.

Using essentially the method of [3], we now prove

**Theorem 4.1.** If \(g \in R_k\), \(\beta > 0\), and \(\lambda \geq 1\), then as \(r \to 1\),
\[
\int_0^{2\pi} |g'(re^{i\theta})|^4 d\theta = O(1) \frac{A(r, g)^{1/2}}{(1 - r)^{\lambda - 1}}.
\]
Proof. Let \( g \in \mathbb{R}_k \) be given by (1.4), and write \( A(r) \) for \( A(r, G) \). Using Theorem 2.2, we see that there exists \( B = B(\beta) \) and \( r_0 < 1 \) such that for \( r > r_0 \), \( M(r^2, g) \leq BA(r^{1/2}) \), and so from Lemma 1(i),

\[
M(r, g) = BA(r)^{1/2}.
\]

However, from Theorem 2.3, \( L(r, g) = O(1)M(r, g) \), and so \( L(r, g) = O(1)A(r)^{1/2} \) as \( r \to 1 \), which gives Theorem 4.1 in the case \( \lambda = 1 \).

We now suppose \( \lambda > 1 \). From (1.5) it follows easily that with \( z = re^{i\theta} \),

\[
\frac{zg'(z)}{g(z)} = \frac{k + 2}{4} p_1(z) - \frac{k - 2}{4} p_2(z),
\]

where \( p_1, p_2 \in \mathcal{P} \), the class of normalized functions with positive real part. Using Minkowski's inequality and well-known properties of the class \( \mathcal{P} \), we find that as \( r \to 1 \),

\[
(1 - r)^{1-\lambda} \int_0^{2\pi} \left| \frac{zg'(z)}{g(z)} \right|^{\frac{\lambda}{2}} d\theta = O(1).
\]

Since

\[
r^{\lambda} \int_0^{2\pi} |g'(z)|^{1/2} d\theta \leq M(r, g) \int_0^{2\pi} \left| \frac{zg'(z)}{g(z)} \right|^{\frac{\lambda}{2}} d\theta,
\]

(4.2) and (4.3) give, as \( r \to 1 \),

\[
r^{\lambda} \int_0^{2\pi} |g'(z)|^{1/2} d\theta = O(1)(1 - r)^{1-\lambda} A(r)^{\lambda/2},
\]

which proves the result.

In conclusion, we examine the same problem for \( V_k \).

Theorem 4.2. If \( f \in V_k \), \( \lambda > 1 \), and \( \beta \geq 1 \), then as \( r \to 1 \),

\[
\int_0^{2\pi} |f'(re^{i\theta})|^{1/\lambda} d\theta = O(1) \left( \frac{A(r, f)^{1/2}}{(1 - r)^{1-1/\lambda}} \right).
\]

Proof. With the notation of Theorem 2.4, \( G(z) - h(z) = ef(z) \), and so with \( z = re^{i\theta} \),

\[
\left( \int_0^{2\pi} |f'(z)|^{1/\lambda} d\theta \right)^{1/\lambda} = O(1) \left( \int_0^{2\pi} |G'(z)|^{1/\lambda} d\theta \right)^{1/\lambda} + O(1)M(r, G).
\]

Since \( G \in \mathbb{R}_K \) and \( \beta = \max\{0, \alpha - 1\} > 0 \), we have from (4.2), (4.4), and
Theorem 4.1 that, as $r \to 1$,

$$\int_0^{2\pi} |f'(z)|^4 \, d\theta = O(1) \frac{A(r, G)^{1/2}}{(1 - r)^{3/4}},$$

and upon using Lemma 1(ii), we find, as $r \to 1$,

$$\int_0^{2\pi} |f'(z)|^4 \, d\theta = O(1) \frac{A(r, f)^{1/2}}{(1 - r)^{3/4}},$$

which proves the theorem.

**Corollary 4.3.** If $f \in V_k$, then as $r \to 1$,

(i) $L(r, f) = O(1) M(r, f)^{1/2}$ if $\alpha > 1$,

(ii) $L(r, f) = O(1) A(r, f)^{1/2} (\log |1 - r|)^{1/2}$ if $\alpha = 1$, and

(iii) $L(r, f)$ and $A(r, f)$ are both bounded if $\alpha < 1$.

**Proof.** (i) Let $\lambda = 1$ in Theorem 4.2.

(ii) From [10] we have $L(r, f) \leq k \pi M(r, f)$. Also,

$$M(r^2, f) \leq \sum_{n=1}^{\infty} |a_n| r^{2n} \leq \left( \sum_{n=1}^{\infty} |a_n|^2 r^{2n} \right)^{1/2} \left( \sum_{n=1}^{\infty} \frac{r^{2n}}{n} \right)^{1/2} \leq A(r, f)^{1/2} \left( \log \frac{1}{1 - r^2} \right)^{1/2}.$$

From Theorem 2.4 and Lemma 1(i), it follows easily that $M(r, f) = O(1) M(r^2, f)$ as $r \to 1$, and hence (ii) is proved. The convex function $f$ given by $f(z) = -\log(1 - z)$, for which $\alpha = 1$, shows that the order of magnitude in (ii) is best possible for all $k \geq 2$.

(iii) If $\alpha < 1$, it follows directly from (1.3) that $f$ is bounded, and hence $L(r, f)$ and $A(r, f)$ are also bounded.

**Bibliography**


E. O. HULBURT CENTER FOR SPACE RESEARCH, U.S. NAVAL RESEARCH LABORATORY, WASHINGTON, D.C. 20390

Current address: College of the Holy Cross, Worcester, Massachusetts 01610