0. Introduction. In [1] the concept of simultaneous approximation and
interpolation which is norm preserving (SAIN) was introduced.

DEFINITION 0.1. Let $X$ be a normed linear space, $M$ a dense subset of $X$, $L$ a finite dimensional subspace of $X^*$. The triple $(X, M, L)$ has property SAIN if for every $x \in X$ and $\varepsilon > 0$ there exists $y \in M$ such that $\|x - y\| < \varepsilon$, $\|x\| = \|y\|$ and $\lambda(x) = \lambda(y)$ for all $\lambda \in L$.

Many examples are given in [1] of various spaces $X$, subsets $M$, and subspaces $L$ such that $(X, M, L)$ is SAIN. In particular [1, Corollary 6.2], let $M$ be the subspace of $l_1$ consisting of the elements having only finitely many nonzero components and let $L = \text{span}\{y_i: i = 1, 2, \ldots, n\} \subseteq l_\infty$. If each $y_i$ is eventually constant, $(y_i = \{y_i(n)\})$ is eventually constant if there is an index $N$ such that $y_i(N) = y_i(N+k)$ for all $k$ then $(l_1, M, L)$ is SAIN.

In [2] a geometrical approach was taken to study the concept of SAIN and in particular it was found that [2, Corollary 3] if $L$ is a finite dimensional subspace of $c_0$ (the predual of $l_1$) $M$ as above, then $(l_1, M, L)$ is SAIN.

It was conjectured that if $L$ is any finite dimensional subspace of $l_\infty$ then $(l_1, M, L)$ is SAIN. In this paper that conjecture is verified.

In this paper the following notation and terminology is used. $X$ denotes a real normed linear space; $X^*$ the continuous dual of $X$, and $U(X)$ and $S(X)$ the closed unit ball and its boundary in $X$. A set $E \subseteq F$ is $F$-extremal, if whenever $tx + (1-t)y \in E$, $0 < t < 1$, and $x, y \in F$ then $x, y \in E$.  

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An extremal subset consisting of a single point is called an extreme point.
The extreme points of a set \( F \) will be denoted \( \text{ext}(F) \). The convex hull of a set \( F \) is the intersection of all convex sets containing \( F \) and will be denoted \( \text{co}(F) \). The closure of a set \( F \) will be denoted \( F \) or \( \text{cl}(F) \).

The real valued function \( \text{sgn}(-) : \text{Reals} \rightarrow \{-1, 0, 1\} \) is defined via \( \text{sgn}(x) = x/|x|, x \neq 0 \) and \( \text{sgn}(0) = 0 \).

1. \( M \)-core.

**Definition 1.1.** Let \( F \) be a set in \( X \), \( M \) a subset of \( F \), then \( F \) is said to have \( M \)-core at \( f \) in \( F \), if given \( m \in M \) there exists \( \delta(m, f) > 0 \) such that \((1 + \delta)f - \delta m \) is in \( F \), for \( 0 \leq \delta \leq \delta(m, f) \). Let \( e_i = \{\delta_i, \} \) denote the usual set of basis vectors in \( l_i \).

**Proposition 1.1.** Let \( x \in S(l_i) \), \( x = \sum_{i \in A} x_i e_i \); \( x_i \neq 0 \), \( i \in A \), \( A \) countable. Let \( F \) be minimal closed \( U(l_i) \) extremal subset containing \( x \). Let \( M = \text{co}(\text{sgn}(x_i)e_i ; i \in A) \). Then \( F \) has \( M \)-core at \( x \).

**Proof.** By [4, Theorem 1] every closed bounded convex subset of \( l_i \) is the closed convex hull of its extreme points. Since the extreme points of \( U(l_i) \) are the usual basis vectors of norm one and \( F \) is \( U(l_i) \) extremal it is easy to see that \( F = \text{cl}(\text{co}(\text{sgn}(x_i)e_i ; i \in A)) \). Assume \( m \) in \( M \) is given. It remains to find \( \delta(x, m) > 0 \) such that \( z = (1 + \delta)x - \delta m \) is in \( F \). Since \( m \) is in \( M \), \( m = \sum_{i \in B} m_i \text{sgn}(x_i)e_i \), \( B \) a finite set. It suffices to find \( \delta \) such that

\[
1 = \|z\| = \sum_{i \in A} (1 + \delta)|x_i| - \delta|m_i| = (1 + \delta) \sum_{i \in A} |x_i| + \sum_{i \in B} |(1 + \delta)x_i - \delta m_i|.
\]

If one can insure that for \( i \in B \), \( \text{sgn}((1 + \delta)x_i - \delta m_i) = \text{sgn}(x_i) \) then one would have

\[
\sum_{i \in B} |(1 + \delta)x_i - \delta m_i| = \sum_{i \in B} [(1 + \delta)x_i - \delta m_i] \text{sgn}(x_i) = \sum_{i \in B} (1 + \delta)x_i \text{sgn}(x_i) - \delta \sum_{i \in B} m_i \text{sgn}(x_i) = (1 + \delta) \sum_{i \in B} x_i - \delta \sum_{i \in B} m_i.
\]

Thus

\[
\|z\| = (1 + \delta) \sum_{i \in A} |x_i| + (1 + \delta) \sum_{i \in B} |x_i| - \delta \sum_{i \in B} |m_i| = (1 + \delta) \|x\| - \delta \|m\| = (1 + \delta) - \delta = 1.
\]

This would conclude the proof. To insure \( \text{sgn}((1 + \delta)x_i - \delta m_i) = \text{sgn}(x_i) \) for all \( i \in B \), let \( \epsilon_i > 0 \) be chosen such that \( |x_i| > \epsilon_i |m_i| \). Let \( \delta = \min_{i \in B} \{\epsilon_i\} \).

Q.E.D.

2. Yamabe's theorem. The following proposition is a version of Yamabe's theorem [1], [4] for convex sets.
Proposition 2.1. Let $X$ be a Banach space, $F$ a convex set contained in $X$. Let $y_1, \ldots, y_n \in X^*$. Let $M$ be a norm dense convex subset of $F$. Let $f \in F$, $F$ having $M$-core at $f$; then given $\varepsilon > 0$ there exists $m \in M$ such that $y_i(f) = y_i(m)$, $i = 1, \ldots, n$ and $\|f - m\| < \varepsilon$.

Proof. Let $K = F \cap \{b : \|b - f\| \leq \varepsilon\}$. $K$ is closed and convex, $N = M \cap K$ is convex and dense in $K$ and $f$ is an $N$ core point for $K$. Define the continuous function $\varphi : F \to \mathbb{R}^n$ via $\varphi(x) = (y_1(x), \ldots, y_n(x))$. Then $\varphi N$ is dense in $\varphi K$ and thus $\varphi N$ contains the relative interior of $\varphi K$. Also $\varphi f$ is a $\varphi N$ core point of $\varphi K$ and this implies that $\varphi f$ is in the relative interior of $\varphi K$, hence in $\varphi N$. It follows that there is an $m \in N$ such that $\varphi m = \varphi f$.

Q.E.D.

3. Application to SAIN in $l_1$.

Definition 3.1. Let $x \in S(X)$; $F(x)$ denotes the minimal closed $U(X)$ extremal subset containing $x$.

Theorem 3.1. Let $X$ be a Banach space, $M$ a dense subspace in $X$. Then $(X, M, L)$ is SAIN for all finite dimensional subspaces $L$ in $X^*$, whenever $F(x) \cap M$ is dense in $F(x)$ and $F(x)$ has $F(x) \cap M$-core at $x$, for every $x \in S(X)$.

Proof. Given $x \in S(X)$, one applies Proposition 2.1 to $F(x)$ and obtains $m \in F(x) \cap M$ satisfying the SAIN conditions at $x$. The homogeneity of the functionals in $L$ and of the norm yields the result for all $x \in X$. Q.E.D.

Theorem 3.2. Let $X = l_1$, $M$ be the subspace of vectors with only finitely many nonzero coordinates. Then for all finite dimensional subspaces $L \subset l_\infty$, $(X, M, L)$ is SAIN.

Proof. Given $x \in S(l_1)$; $F(x)$, $F(x) \cap M$ satisfy the hypotheses of Proposition 1.1. Hence, they satisfy the hypotheses of Theorem 3.1. Q.E.D.

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References


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