

FUBINI THEOREMS FOR ORLICZ SPACES OF LEBESGUE-BOCHNER MEASURABLE FUNCTIONS

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ABSTRACT. Let (X, V, v) be the volume space formed as the product of the volume spaces (X_i, V_i, v_i) ($i=1, 2$). Let (p, q) be a pair of complementary (continuous) Young's functions, let Y, Z, Z_1, Z_2, W be Banach spaces, let w be a multilinear continuous operator on $Y \times Z_1 \times Z_2 \rightarrow W$. Let $L_p(v, Y)$ be the Orlicz space of Lebesgue-Bochner measurable functions generated by p , and let $K_q(v, Z)$ be the associated space of finitely additive Z -valued set functions. The principal result of this paper is as follows: Let $f \in L_p(v, Y)$, $\mu_2 \in K_q(v_2, Z_2)$. Then (a) the function $f(x_1, \cdot)$ is v_2 -Orlicz summable v_1 -a.e.; (b) the operator $r(f, \mu_2)$ defined by the expression $r(f, \mu_2)(x_1) = \int w_1(f(x_1, x_2), \mu_2(dx_2))$ v_1 -a.e. is bilinear and continuous from $L_p(v, Y) \times K_q(v_2, Z_2)$ into $L_p(v_1, Y_1)/N$, where $w_1(y, z_2) = w(y, z_2)$, where Y_1 is the Banach space of bounded linear operators from Z_1 into W , and where N is the set of Y_1 -valued v_1 -measurable functions of zero seminorm; (c) the equality $\int w(f, d\mu_1, d\mu_2) = \int w_0(r(f, \mu_2), d\mu_1)$ holds for all $f \in L_p(v, Y)$, $\mu_i \in K_q(v_i, Z_i)$ ($i=1, 2$), where $w_0(y_1, z_1) = y_1(z_1)$ for all $y_1 \in Y_1$, $z_1 \in Z_1$.

The purpose of this paper is to complete the basic integration theory for the Orlicz spaces of Bochner measurable functions. In establishing the initial part of this theory of Orlicz spaces, Diestel [4] established the following results: a characterization of convergence, a monotone convergence theorem, a dominated convergence theorem, and the density of simple functions in the Orlicz space. The Fubini type results of this paper essentially complete the basic theory of Orlicz spaces of Bochner summable functions.

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The Orlicz spaces treated in this paper deviate from the classical Orlicz spaces in that, instead of their being generated from the usual measure space, we shall generate them from a volume space and from the ensuing integration theory of Bogdanowicz [1]. The volume space is a natural environment for Fubini type theorems. That it is also a valid environment is established by the work of Diestel [4] and by the extensive work of Bogdanowicz (see [3] for a suitable bibliography). The notions of volume and prering are not alien to integration theory, since a family of measurable sets on which a measure is finite is a prering and the measure restricted to this domain is a volume.

1. Preliminaries. This paper will follow the notation of Diestel [4]. The principal deviation from standard Orlicz space notation is that we will use the letters p, q for the complementary Young's functions. Abbreviations of the form Th. 1, Th. 3 and so on, will refer to the corresponding results in reference [4]. The underlying integration theory we shall assume in this paper is the theory developed by W. Bogdanowicz [1].

Let V be a family of subsets of an abstract set X such that for any two sets A, B from V we have $A \cap B \in V$ and $A - B$ is a finite union of disjoint sets from V . The family V is called a prering.

A nonnegative, real-valued function v defined on the prering V is called a volume if for every countable family of disjoint sets $A_t \in V$ ($t \in T$) such that $A = \bigcup_{t \in T} A_t \in V$ we have $v(A) = \sum_T v(A_t)$.

The triple (X, V, v) is called a volume space. For Y a Banach space the space $S(V, Y)$ of V -simple Y -valued functions and the space $L(v, Y)$ of Lebesgue-Bochner summable functions are defined in Bogdanowicz [1], as are the notions of v -null sets and v -a.e. The space $M(v, Y)$ of Y -valued v -measurable functions is defined in Bogdanowicz [2].

Let (p, q) be a pair of nonnegative, real-valued functions defined on the interval $[0, \infty)$ such that p is continuous from $(0, \infty)$ onto $(0, \infty)$ and differentiable with derivative p' on $(0, \infty)$, p' is a homeomorphism of $(0, \infty)$ with itself such that if $(p')^{-1}$ denotes the inverse function of p' then

$$q(x) = \int_0^x (p')^{-1}(t) dt.$$

Then (p, q) are called complementary Young's functions.

We say that the function p satisfies the Δ_2 -condition if there exists a constant $M > 0$ such that $p(2x) \leq Mp(x)$ for all nonnegative x . We say that the function p satisfies the Δ' -condition if there exists a constant $k > 0$ such that $p(xy) \leq kp(x)p(y)$ for all nonnegative x and y .

Let Y, Z, W be Banach spaces and let U be the space of all bilinear continuous operators on the space $Y \times Z$ into the space W . Denote the

norms of the above spaces by $|\cdot|$. Assume that $u \in U$. Let (X, V, v) be a volume space and let (p, q) be a pair of complementary Young's functions.

Let $L_p(v, Y)$ be the space of functions f mapping the space X into the space Y which satisfy the following conditions:

- (i) f is v -locally summable (that is, $c_A f \in L(v, Y)$ for all $A \in V$);
- (ii) f has a support from V_σ (that is, there exist $A_i \in V$, $i=1, 2, \dots$, such that $f(x)=0$ if $x \notin \bigcup_i A_i$);
- (iii) $\|f\|_{p,v} < \infty$, where the functional $\|f\|_{p,v}$ is defined by the formula

$$\|f\|_{p,v} = \sup \left\{ \int |f| s \, dv : s \in Q(v, R^+) \right\},$$

where $Q(v, R^+)$, which shall be called the norming set for the space $L_p(v, Y)$, is the set of all nonnegative V -simple functions s such that $\int q \circ s \, dv \leq 1$.

According to Th. 2 the space $(L_p(v, Y), \|\cdot\|_{p,v})$ is a complete seminormed linear space. If it is assumed that the Young's function p satisfies the Δ_2 -condition then the space $S(V, Y)$ of simple functions is dense in the space $L_p(v, Y)$. If we define the functional $\|\cdot\|_p$ on the space $L_p(v, Y)$ by the expression

$$\|\cdot\|_p = \inf \left\{ k > 0 : \int p(f/k) \, dv \leq 1 \right\}$$

then it follows easily from Young's inequality and from Th. 4 that $\|\cdot\|_p$ is a seminorm on $L_p(v, Y)$ which is equivalent to the seminorm $\|\cdot\|_{p,v}$.

2. Spaces of finitely additive set functions. In this section we shall discuss two spaces of finitely additive set functions which are related to the Orlicz space $L_p(v, Y)$, one space of which has two equivalent formulations.

For Z a Banach space and $u \in U$ let $K_{q,u}(v, Z)$ be the space of all finitely additive functions μ mapping the prering V into the Banach space Z such that $\mu(A)=0$ if $v(A)=0$ and the functional $\|\mu\|_{q,u}$ is finite, where

$$\|\mu\|_{q,u} = \sup \left\{ \left\| \sum_i u(y_i, \mu(A_i)) \right\| : y_i \in Y, \{A_i\} \text{ is a finite family of disjoint sets from } V \text{ and } \left\| \sum_i y_i c_{A_i} \right\|_{p,v} \leq 1 \right\}.$$

It is clear that the space $K_{q,u}(v, Z)$ is a seminormed space.

Let $K_q(v, Z)$ be the space of finitely additive functions μ mapping the prering V into the Banach space Z such that $\mu(A)=0$ if $v(A)=0$ and the

functional $\|\mu\|_q$ is finite, where

$$\|\mu\|_q = \sup \left\{ \sum_i a_i |\mu(A_i)| : a_i \geq 0, \{A_i\} \text{ is a finite family} \right. \\ \left. \text{of disjoint sets from } V \text{ and } \left\| \sum_i a_i c_{A_i} \right\|_{p,v} \leq 1 \right\}.$$

We notice that the space $K_q(v, Z)$ is a subset of the space $K_{q,u}(v, Z)$ and $\|\mu\|_{q,u} \leq \|u\| \|\mu\|_q$ for all $\mu \in K_q(v, Z)$.

Denote by $A^q(v, Z)$ the space of all finitely additive functions μ from the prepring V into the space Z which satisfy the following conditions: (i) $\mu(A) = 0$ when $v(A) = 0$, for $A \in V$, (ii) $I_q(\mu/k) \leq 1$ for some positive k , where the function I_q is defined by the following expression:

$$I_q(\mu) = \sup \sum \{q(|\mu(A)|/v(A))v(A) : A \in \mathcal{F}\}$$

where the supremum is taken over all finite families \mathcal{F} of disjoint sets from the prepring V (we observe the convention $0/0 = 0$ here and throughout). By Theorem 11 of Uhl [7] the functional N_q defined by

$$N_q(\mu) = \inf \{k > 0 : I_q(\mu/k) \leq 1\}$$

is a norm under which the space $A^q(v, Z)$ is a Banach space.

PROPOSITION 1. *The sets $K_q(v, Z)$ and $A^q(v, Z)$ are equal, and the respective norms satisfy the following inequality:*

$$N_q(\mu) \leq \|\mu\|_q \leq 2N_q(\mu) \quad \text{for all } \mu \in K_q(v, Z).$$

PROOF. The proof of Lemma 7 of Rao [6, p. 563] applies here, with only the obvious changes needed.

It follows from Proposition 1 that the space $K_q(v, Z)$ is a Banach space.

Let $s \in S(V, Y)$ and $\mu \in K_{q,u}(v, Z)$. Define the integral $\int u(s, d\mu)$ by the formula

$$\int u(s, d\mu) = \sum_i u(y_i, \mu(A_i)),$$

where $s = \sum_i y_i c_{A_i}$ with $y_i \in Y$ and $\{A_i\}$ a finite family of disjoint sets from V . One can prove that $\int u(s, d\mu)$ is well defined and that the operator $\int u(\cdot, \cdot)$ is bilinear and continuous from the space $S(V, Y) \times K_{q,u}(v, Z)$ into the space W and satisfies the relationship

$$\left| \int u(s, d\mu) \right| \leq \|s\|_{p,v} \|\mu\|_{q,u}.$$

If we assume that the Young's function p satisfies the Δ_2 -condition then from Th. 8 we may extend the operator $\int u(\cdot, \cdot)$ onto the space $L_p(v, Y) \times K_{q,u}(v, Z)$. The extended operator is bilinear continuous on this space and

satisfies the relationship

$$\left| \int u(f, d\mu) \right| \leq \|f\|_{p,v} \|\mu\|_{q,u} \quad \text{for } f \in L_p(v, Y), \mu \in K_{q,u}(v, Z).$$

3. Products of vector-valued set functions. For the volume spaces (X_i, V_i, v_i) ($i=1, \dots, n$) define the triple (X, V, v) as follows: $X = X_1 \times \dots \times X_n$, V is the family of all sets of the form $A_1 \times \dots \times A_n$ where $A_i \in V_i$, and $v(A) = v_1(A_1) \dots v_n(A_n)$ for $A \in V$. It is clear that (X, V, v) is a volume space.

Let the volume space (X, V, v) be the product of the volume spaces (X_i, V_i, v_i) ($i=1, 2$).

THEOREM 2. Let $\mu \in K_q(v, Z)$ and $A \in V_1$ with $v_1(A) > 0$. If we define the functions μ_A, v_A by the formulae

$$\mu_A(B) = \mu(A \times B), \quad v_A(B) = v(A \times B) \quad \text{for all } B \in V_2$$

then (i) v_A is a volume on the prepring V_2 , (ii) the spaces $K_q(v_A, Z)$ and $K_q(v_2, Z)$ are equal, and (iii) $\mu_A \in K_q(v_2, Z)$.

PROOF. Let $A \in V_1$. It is clear that v_A is a volume on the prepring V_2 . Put $c = v_1(A)$ and assume $c > 0$. That the sets $K_q(v_A, Z)$ and $K_q(v_2, Z)$ are equal will follow from Proposition 1 once we have shown that the sets $A^q(v_A, Z)$ and $A^q(v_2, Z)$ are equal.

Let \mathcal{F} be a finite family of disjoint sets from the prepring V_2 . Let $\mu \in A^q(v_2, Z)$, and let $k > 0$ be such that $I_q(\mu/k)_{v_2} \leq 1$. Then we have the following relations:

$$\begin{aligned} & \sum \{q[c|\mu(B)|/kv_A(B)]v_A(B) : B \in \mathcal{F}\} \\ &= \sum \{q[|\mu(B)|/kv_2(B)]v_2(B) : B \in \mathcal{F}\} \leq 1. \end{aligned}$$

If $c > 1$ then since $cq(x) \leq q(cx)$ for all $x \geq 0$ we have

$$\sum \{q[|\mu(B)|/kv_A(B)]v_A(B) : B \in \mathcal{F}\} \leq 1.$$

If $c \leq 1$ then

$$\sum \{q[|\mu(B)|/(k/c)v_A(B)]v_A(B) : B \in \mathcal{F}\} \leq c \leq 1.$$

Thus if $e = \min(c, 1)$ then $eN_q(\mu)_{v_A} \leq N_q(\mu)_{v_2}$ and $\mu \in A^q(v_A, Z)$. Similarly one can show that $A^q(v_A, Z)$ is a subset of $A^q(v_2, Z)$ and that if $f = \max(c, 1)$ then $N_q(\mu)_{v_2} \leq fN_q(\mu)_{v_A}$. To show that $\mu_A \in K_q(v_2, Z)$ we need only prove that $\mu_A \in A^q(v_A, Z)$, which follows from definitions, Proposition 1 and (ii) above.

THEOREM 3. Let $\mu \in K_{q,u}(v, Z)$ and $A \in V_1$. If the functions μ_A, v_A are defined as in Theorem 2 then $\mu_A \in K_{q,u}(v_2, Z)$ and $\|\mu_A\|_{q,u} \leq (1 + v_1(A))\|\mu\|_{q,u}$ for all $A \in V_1$.

PROOF. It is clear that the function μ_A is finitely additive on the pre-ring V_2 and that $\mu_A(B)=0$ when $v_2(B)=0$. Let $s=\sum_i y_i c_{A_i}$ be such that $\|s\|_{p,v_2} \leq 1$. Then since $s \cdot c_A$ is a simple function $s \cdot c_A \in L_p(v, Y)$. From Young's inequality we have $\|s \cdot c_A\|_{p,v} \leq 1 + \int p \circ |s \cdot c_A| dv$, and from Theorem 3 of Bogdanowicz [3] we have the relation $\int p \circ |s \cdot c_A| dv = v_1(A) \int p \circ |s| dv_2$. From Th. 4 we have the estimate $\int p \circ |s| dv_2 \leq 1$, so that $\|s \cdot c_A\|_{p,v} \leq a$, where $a=1+v_1(A)$. Thus we have

$$\left| \sum_i u(y_i/a, \mu_A(A_{2i})) \right| \leq \|\mu\|_{q,u},$$

so that $\|\mu_A\|_{q,u} \leq a \|\mu\|_{q,u}$ and $\mu_A \in K_{q,u}(v_2, Z)$.

We remark that Theorem 2 could also be proved in a manner analogous to the proof of Theorem 3, the proof of which would yield an inequality similar to that of Theorem 3.

Let Z_i ($i=1, \dots, n$), W be Banach spaces and let u be an n -linear continuous operator from the product of the spaces Z_1, \dots, Z_n into the space W . Let $B(Z_1, \dots, Z_n; W)$ denote the space of all such operators u . Denote the norms in the above spaces by $|\cdot|$. Let (X, V, v) be the product volume space formed from the volume spaces (X_i, V_i, v_i) ($i=1, \dots, n$).

THEOREM 4. *Assume that the Young's function q satisfies the Δ' -condition. If $\mu_i \in K_q(v_i, Z_i)$ for $i=1, \dots, n$ then $\mu \in K_q(v, W)$, where*

$$\mu(A_1 \times \dots \times A_n) = u(\mu_1(A_1), \dots, \mu_n(A_n)) \quad \text{for } A \in V.$$

PROOF. The proof is given in [9] for Orlicz spaces of finitely additive set functions generated from a charge space. The proof there is applicable here.

4. A Fubini theorem for the Orlicz space $L_p(v, Y)$. Let (X, V, v) be the product volume space of the volume spaces (X_i, V_i, v_i) for $i=1, 2$. Let Y, Z_1, Z_2, W be Banach spaces. Assume that $w \in B(Y, Z_1, Z_2; W)$ and define the operator w_1 by $w_1(y, z_2) = w(y, \cdot, z_2)$ for $y \in Y, z_2 \in Z_2$. It is clear that $w_1 \in B(Y, Z_2; Y_1)$, where $Y_1 = B(Z_1; W)$. Define $w_0(y_1, z_1) = y_1(z_1)$ for $y_1 \in Y_1, z_1 \in Z_1$ and notice that $w_0 \in B(Y_1, Z_1; W)$ and $|w| = |w_1|$ and $|w_0| = 1$.

Let $N(v_1, Y_1)$ denote the subset of those functions from the space $L_p(v_1, Y_1)$ whose seminorms are zero. The space $N(v_1, Y_1)$ is a linear space and the quotient space $L_p(V_1, Y_1)/N(v_1, Y_1)$ is a Banach space with norm $\|\cdot\|_{p,v_1}$ defined by the formula

$$\|[f]\|_{p,v_1} = \|f\|_{p,v_1} \quad \text{for all } f \in L_p(v_1, Y_1).$$

Let us assume that the Young's functions p, q satisfy respectively the Δ_2 -condition and the Δ' -condition, with k the associated Δ' -constant.

THEOREM 5. *Let $f \in L_p(v, Y)$ and $\mu_2 \in K_q(v_2, Z_2)$. Then the following statements are true:*

- (i) *There exists a v_1 -null set D such that $f(x_1, \cdot) \in L_p(v_2, Y)$ if $x_1 \notin D$.*
- (ii) *If the function t is defined by the formula $t(x_1) = \|f(x_1, \cdot)\|_{p, v_2}$ if $x_1 \notin D$ and 0 otherwise then $t \in L_p(v_1, R)$ and t satisfies the inequality $\|t\|_{p, v_1} \leq \int p \circ |f| dv + k$.*
- (iii) *The operator $\tilde{f} = r(f, \mu_2)$ defined by the formula*

$$\tilde{f}(x_1) = \int w_1(f(x_1, x_2), \mu_2(dx_2))$$

if $x_1 \notin D$ is bilinear from the space $L_p(v, Y) \times K_q(v_2, Z_2)$ into the space $L_p(v_1, Y_1)/N(v_1, Y_1)$ and satisfies the inequality $\|\tilde{f}\|_{p, v_1} \leq \|\mu_2\|_q \|t\|_{p, v_1} |w|$.

- (iv) $\int w(f, d\mu_1, d\mu_2) = \int w_0(r(f, \mu_2), d\mu_1)$ for all $f \in L_p(v, Y)$, $\mu_i \in K_q(v_i, Z_i)$ for $i=1, 2$.

PROOF. Let $f \in L_p(v, Y)$. By Th. 1 and Tonelli's theorem there is a v_1 -null set D_1 such that $f(x_1, \cdot) \in M(v_2, Y)$ if $x_1 \notin D_1$. By the corollary to Th. 4 and by Theorem 3 of [3] there is a v_1 -null set D_2 such that $p \circ |f(x_1, \cdot)| \in L(v_2, R)$ if $x_1 \notin D_2$. Put $D = D_1 \cup D_2$ and notice that D is a v -null set. Let $s \in S(V_2, R)$ be such that $\|s\|_{p, v_2} \leq 1$ and assume that $x_1 \notin D$. Then we get the estimate

$$\int |f(x_1, x_2)| s(x_2) v_2(dx_2) \leq \int p(|f(x_1, x_2)|) v_2(dx_2) + 1,$$

so that

$$\|f(x_1, \cdot)\|_{p, v_2} \leq \int p(|f(x_1, x_2)|) v_2(dx_2) + 1.$$

It follows from Th. 1 that $f(x_1, \cdot) \in L_p(v_2, Y)$ if $x_1 \notin D$.

For the operator \tilde{f} , as defined in the theorem statement, put $\tilde{f}(x_1) = 0 \in Y_1$ if $x_1 \in D$. We shall first show that the function $\tilde{f} \in M(v_1, Y_1)$.

Define the function b_m by the formula

$$\begin{aligned} b_m(y) &= y & \text{if } |y| \leq m, \\ &= (m/|y|)y & \text{if } |y| > m, \end{aligned}$$

for all $y \in Y$ and for $m=1, 2, \dots$

Since $f \in M(v, Y)$ there is a sequence of functions $r_n \in S(V, Y)$ and a v -null set A such that $r_n(x) \rightarrow f(x)$ if $x \notin A$. Let C be the v -null set such that the section $A(x_1) = \{x_2 : (x_1, x_2) \in A\}$ is a v_2 -null set if $x_1 \notin C$. Let the set A_n be the support of the function r_n and put $B_n = A_1 \cup \dots \cup A_n$, for $n=1, 2, \dots$

If for an appropriate strictly increasing sequence of positive integers $m(n)$ we define the function s_n by the formula $s_n(x) = c_{B_n}(x)b_n(r_{m(n)}(x))$ for all $x \in X$, for $n = 1, 2, \dots$, then we get $\|f(x_1, \cdot) - s_n(x_1, \cdot)\|_{p, v_2} \rightarrow 0$ if $x_1 \notin D \cup C$. Define the function \bar{s}_n by the formula $\bar{s}_n(x_1) = \int w_1(s_n(x_1, x_2), \mu_2(dx_2))$ for all $x_1 \in X_1$, for $n = 1, 2, \dots$. We have the estimate

$$|\bar{s}_n(x_1) - \bar{f}(x_1)| \leq \|s_n(x_1, \cdot) - f(x_1, \cdot)\|_{p, v_2} \|\mu_2\|_q |w|$$

if $x_1 \notin D \cup C$. Thus $\bar{s}_n \rightarrow \bar{f}$ v_1 -a.e., and since $\bar{s}_n \in S(V_1, Y_1)$ it follows from Theorem 1 of [2] that $\bar{f} \in M(v_1, Y_1)$.

Let $s \in Q(v_2, R^+)$ and $s_0 \in Q(v_1, R^+)$. From Young's inequality and the fact that q satisfies the Δ' -condition we get the following inequality:

$$\int |f(x_1, x_2)| s(x_2) v_2(dx_2) s_0(x_1) \leq \int p(|f(x_1, x_2)|) v_2(dx_2) + kq(s_0(x_1)) \quad v_1\text{-a.e.}$$

If we define the function t on X_1 by $t(x_1) = \|f(x_1, \cdot)\|_{p, v_2}$ if $x_1 \notin D$, and 0 otherwise then we get the estimates

$$|\bar{f}(x_1)| s_0(x_1) \leq t(x_1) \|\mu_2\|_q |w| s_0(x_1)$$

and

$$t(x_1) s_0(x_1) \leq \int p(|f(x_1, x_2)|) v_2(dx_2) + kq(s_0(x_1)) \quad v_1\text{-a.e.}$$

If we temporarily assume that the function t is v_1 -measurable then by integrating the last two inequalities and by taking the appropriate suprema we have the following two inequalities:

$$\|\bar{f}\|_{p, v_1} \leq \|t\|_{p, v_1} \|\mu_2\|_q |w|, \quad \|t\|_{p, v_1} \leq \int p \circ |f| dv + k.$$

Thus $\bar{f} \in L_p(v_1, Y_1)/N(v_1, Y_1)$. The bilinearity of the operator r is evident.

We next show that the function t is locally v_1 -summable and that t has a $(V_1)_\sigma$ support. The latter is true because $f \in L_p(v, Y)$. Fix $B \in V_1$ for the remainder of this discussion and choose $s \in Q(v_2, R^+)$. Notice that $|f(x_1, \cdot)| s(\cdot) \in L(v_2, R)$ if $x_1 \notin D$. If we put $g_s(x_1) = c_B(x_1) \int |f(x_1, x_2)| s(x_2) v_2(dx_2)$ if $x_1 \notin D$ and 0 otherwise then from Theorem 3 of [3] we have that $g_s \in L(v_1, R)$. Define the function h by the formula

$$h(x_1) = c_B(x_1) \left[1 + \int p \circ |f(x_1, x_2)| v_2(dx_2) \right]$$

if $x_1 \notin D$ and 0 otherwise, where $p \circ |f(x_1, \cdot)|$ is v_2 -summable by Th. 4. From Theorem 3 of [3] we have that $h \in L(v_1, R)$. From Young's inequality we have $g_s(x_1) \leq h(x_1)$ for all x_1 . Let $G = \{g_s : s \in Q(v_2, R^+)\}$ then G is a bounded set from $L(v_1, R)$. Since $L(v_1, R)$ is order complete with respect to pointwise domination we have that the function $tc_B = \sup\{g_s : s \in Q(v_2, R^+)\}$ is v_1 -summable.

Part (iv) of the theorem can easily be shown true for simple functions. That it is true for functions from $L_p(v, Y)$ follows from the density of simple functions in $L_p(v, Y)$ (Th. 8) and from the boundedness and coordinate-wise linearity of the integral.

REMARK. The statement derived from Theorem 5, excluding part (iv), by replacing the space $K_q(v_2, Z_2)$ by the space $K_{q,w}(v_2, Z_2)$ and by replacing the value $\|\mu_2\|_q|w|$ by the value $\|\mu_2\|_{q,w}$ is true. The proof is similar to that of Theorem 5.

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