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AN ELEMENTARY PROOF OF A RADON-NIKODÝM THEOREM FOR FINITELY ADDITIVE SET FUNCTIONS

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ABSTRACT. In 1967 Charles Fefferman proved a Radon-Nikodým theorem for finitely additive measures. We give an elementary proof of a generalization of this theorem.

In this paper we give an elementary proof of the

THEOREM. *Let μ, γ be real-valued bounded finitely additive set functions on an algebra Σ of subsets of the set S. Then for every $\varepsilon>0$ there exists $N\in\Sigma$ and a μ-simple function f on S such that*

$$E \in \Sigma \;\&\; E \subseteq N \Rightarrow |\mu E| < \varepsilon,$$

$$E \in \Sigma \;\&\; E \subseteq S\backslash N \Rightarrow \left|\gamma E - \int_E f\, d\mu\right| < \varepsilon.$$

(*Hence there is a μ-simple function g with $E\in\Sigma\Rightarrow|\gamma E-\gamma(E\cap N)-\int_E g d\mu|<\varepsilon$.*)

If γ is absolutely μ-continuous (i.e., for every $\varepsilon>0$ there exists $\delta>0$ such that $|\mu E|<\delta\Rightarrow|\gamma E|<\varepsilon$), the Theorem reduces to a result of Fefferman which he proved using functional analysis (see Corollary).

LEMMA 1 ("WEAK HAHN DECOMPOSITION"). *Let ν be a real-valued bounded finitely additive set function on an algebra $\Sigma\subseteq\exp S$. For every $\alpha>0$ there is a $B\in\Sigma$ (written $B(\nu, \alpha)$) such that*

$$E \in \Sigma \;\&\; E \subseteq B \Rightarrow \nu E > -\alpha, \qquad E \in \Sigma \;\&\; E \subseteq S\backslash B \Rightarrow \nu E < \alpha.$$

PROOF. ν is bounded and so there is a smallest natural number k with $\nu\leqq k\alpha$; for some $B\in\Sigma$ we have $\nu B>k\alpha-\alpha$. If $E\in\Sigma$ and $E\subseteq B$, then $\nu E=\nu B-\nu(B\backslash E)>k\alpha-\alpha-k\alpha=-\alpha$; if $E\in\Sigma$ and $E\subseteq S\backslash B$, then $\nu E=\nu(B\cup E)-\nu B<k\alpha-(k\alpha-\alpha)=\alpha$.

LEMMA 2 (CF. [**2**, §22.2], [**1**, THEOREM 2]). *Let μ, γ be finite real-valued finitely additive set functions on an algebra $\Sigma\subseteq\exp S$. If there exists $\delta>0$ such that $\mu>-\delta/2$ and $\gamma>-\delta/2$, then there exist $N\in\Sigma$ and a μ-simple function f on S such that*

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$$E \in \Sigma \ \& \ E \subseteq N \Rightarrow |\mu E| < \delta,$$
$$E \in \Sigma \ \& \ E \subseteq S \backslash N \Rightarrow \left|\gamma E - \int_E f\, d\mu\right| < \delta.$$

PROOF. Let m be an integer $>2(\gamma S+\delta)\cdot(\mu S+\delta)/\delta^2$. Put

$$B_k = B\left(\gamma - \frac{\delta k}{2(\mu S + \delta)}\mu, \frac{\delta}{2m}\right) \qquad \text{(by Lemma 1)},$$
$$A_1 = S \backslash \bigcup_{j=1}^{m} B_j, \quad A_k = B_{k-1} \backslash \bigcup_{j=k}^{m} B_j, \qquad k = 2, 3, \cdots, m,$$
$$N = B_m, \qquad f = \sum_{k=1}^{m} \frac{\delta(k-1)}{2(\mu S + \delta)} \chi_{A_k},$$

where χ_A is the characteristic function of A. If $E \in \Sigma$ and $E \subseteq N$, then

$$\gamma E - \frac{\delta m}{2(\mu S + \delta)}\mu E > -\frac{\delta}{2m} \geqq -\frac{\delta}{2};$$

thus

$$\mu E < \frac{2}{m}\frac{\mu S + \delta}{\delta}\left(\gamma E + \frac{\delta}{2}\right) < \frac{\delta^2}{(\gamma S + \delta)(\mu S + \delta)}\frac{\mu S + \delta}{\delta}$$
$$\times \left(\gamma S - \gamma(S \backslash E) + \frac{\delta}{2}\right) < \frac{\delta}{\gamma S + \delta}\left(\gamma S + \frac{\delta}{2} + \frac{\delta}{2}\right) = \delta.$$

If $E \in \Sigma$ and $E \subseteq S \backslash N$, then E is the disjoint union of the sets $E_1, \cdots, E_m$, where $E_k = E \cap A_k$ for $k = 1, 2, \cdots, m$. Moreover, since $E_k \subseteq A_k \subseteq S \backslash B_k$, we have

$$\gamma E_k - \frac{\delta k}{2(\mu S + \delta)}\mu E_k < \frac{\delta}{2m} \quad \text{for } k = 1, 2, \cdots, m,$$

and since $E_k \subseteq A_k \subseteq B_{k-1}$, we have

$$\gamma E_k - \frac{\delta(k-1)}{2(\mu S + \delta)}\mu E_k > -\frac{\delta}{2m} \quad \text{for } k = 2, \cdots, m.$$

Hence

$$-\delta < -\frac{\delta}{2} + (m-1)\left(-\frac{\delta}{2m}\right) < \sum_{k=1}^{m}\left(\gamma E_k - \frac{\delta(k-1)}{2(\mu S + \delta)}\mu E_k\right)$$
$$= \sum_{k=1}^{m}\left(\gamma E_k - \frac{\delta k}{2(\mu S + \delta)}\mu E_k\right) + \sum_{k=1}^{m}\frac{\delta}{2(\mu S + \delta)}\mu E_k$$
$$< m\frac{\delta}{2m} + \frac{\delta}{2}\frac{\mu E}{\mu S + \delta} < \delta.$$

Since $\sum_{k=1}^{m}(\gamma E_k-(\delta(k-1)/2(\mu S+\delta))\mu E_k)=\gamma E-\int_E f\,d\mu$, Lemma 2 is proved.

PROOF OF THE THEOREM. Let $B=B(\mu,\varepsilon/8)$, $B'=B(\gamma,\varepsilon/8)$, $\mu_1E=\mu(E\cap B)$, $\mu_2E=-\mu(E\backslash B)$, $\gamma_1E=\gamma(E\cap B')$ and $\gamma_2E=-\gamma(E\backslash B')$ for $E\in\Sigma$ (so that $\mu=\mu_1-\mu_2$ and $\gamma=\gamma_1-\gamma_2$). Clearly $\mu_i>-\varepsilon/8$ and $\gamma_i>\varepsilon/8$ for $i=1,2$. By Lemma 2 (with $\delta=\varepsilon/4$) for $i,j=1,2$ there are sets N_{ij} and μ-simple functions f_{ij} such that

$$E\in\Sigma\ \&\ E\subseteq N_{ij}\Rightarrow|\mu_iE|<\frac{\varepsilon}{4},$$

$$E\in\Sigma\ \&\ E\subseteq S\backslash N_{ij}\Rightarrow\left|\gamma_jE-\int_E f_{ij}\,d\mu_i\right|<\frac{\varepsilon}{4}.$$

Let $N=\bigcup_{i,j=1,2}N_{ij}$ and $f=(f_{11}-f_{12})\chi_B+(f_{21}-f_{22})\chi_{S\backslash B}$. If $E\in\Sigma$ and $E\subseteq N$, then E is the disjoint union of sets E_{ij}, $i,j=1,2$, with $E_{ij}\in\Sigma$ and $E_{ij}\subseteq N_{ij}$,

$$\mu E=\sum_{i,j=1,2}\mu E_{ij}\quad\text{and}\quad-\varepsilon<4\cdot\left(-\frac{\varepsilon}{8}\right)<\sum_{i,j=1,2}\mu E_{ij}<4\cdot\frac{\varepsilon}{4}=\varepsilon,$$

i.e., $|\mu E|<\varepsilon$. If $E\in\Sigma$ and $E\subseteq S\backslash N$, then $E\subseteq S\backslash N_{ij}$ for $i,j=1,2$; in addition $\mu_2(E\cap B)=\mu_1(E\backslash B)=0$, and thus

$$\begin{aligned}\left|\gamma E-\int_E f\,d\mu\right|&\leqq\left|\gamma(E\cap B)-\int_{E\cap B}(f_{11}-f_{12})\,d\mu\right|\\&\quad+\left|\gamma(E\backslash B)-\int_{E\backslash B}(f_{21}-f_{22})\,d\mu\right|\\&\leqq\left|\gamma_1(E\cap B)-\int_{E\cap B}f_{11}\,d\mu_1\right|\\&\quad+\left|\gamma_2(E\cap B)-\int_{E\cap B}f_{12}\,d\mu_1\right|\\&\quad+\left|\gamma_1(E\backslash B)-\int_{E\backslash B}f_{21}\,d\mu_2\right|\\&\quad+\left|\gamma_2(E\backslash B)-\int_{E\backslash B}f_{22}\,d\mu_2\right|<4\cdot\frac{\varepsilon}{4}=\varepsilon.\end{aligned}$$

The Theorem is proved.

It is now easy to prove the

COROLLARY. *Let μ, γ be bounded real-valued finitely additive set functions on an algebra $\Sigma\subseteq\exp S$. Let γ be absolutely μ-continuous (in the*

sense mentioned above). Then for every $\varepsilon>0$ there is a μ-simple function f such that

$$E \in \Sigma \Rightarrow \left|\gamma E - \int_E f\, d\mu\right| < \varepsilon.$$

As Fefferman [**1**, Lemma 1] has shown, it suffices to suppose that γ is finite (instead of bounded).

By the Corollary (if μ and γ satisfy its hypotheses), there is a sequence $\{f_n\}$ of μ-simple functions f_n with $\lim_{n\to\infty} \int_E f_n\, d\mu = \gamma E$ uniformly for $E \in \Sigma$ (it was in this form formulated in [**1**]).

The following example shows there need not be an increasing such sequence:

S is the set of all natural numbers ($\geqq 1$),

Σ is the algebra of all finite subsets of S and their complements,

$$\gamma E = 0, \qquad \mu E = \sum_{k\in E} 2^{-k} \qquad \text{for finite } E,$$

$$\gamma E = 1, \qquad \mu E = 1 + \sum_{k\in E} 2^{-k} \quad \text{for infinite } E.$$

References

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2. Eduard Čech, *Point sets*, Jednota Československ. Mat. a Fyz., Prague, 1936. (Czech).

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