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THE WIENER CLOSURE THEOREMS FOR ABSTRACT WIENER SPACES[1]

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ABSTRACT. We introduce $\mathscr{L}_1$ and $\mathscr{L}_2$ translates for functions in $\mathscr{L}_1(\mu)$ and $\mathscr{L}_2(\mu)$ where μ is a Gaussian measure on a Banach space. With these translates and the Fourier-Wiener transforms defined by Cameron and Martin we obtain Wiener's closure theorem in $\mathscr{L}_2(\mu)$ and in $\mathscr{L}_1(\mu)$. Using the $\mathscr{L}_1(\mu)$ results we indicate the analogue of the Wiener-Pitt Tauberian theorems for this setup.

1. **Introduction.** Let μ be the Wiener measure on $C[0, 1]$ and $\mathscr{L}_2(\mu)$ be the space of square integrable Borel functions with respect to μ. For $f \in \mathscr{L}_2(\mu)$, the Fourier-Wiener transform was defined by Cameron and Martin [**3**]. In this paper we extend this notion to abstract Wiener spaces [**6**] and obtain an analogue of Wiener's closure theorem [**10**] for $\mathscr{L}_2(\mu)$. Our main effort however is to obtain an analogue of Wiener's closure theorem for $\mathscr{L}_1(\mu)$. From this theorem one can easily derive the Wiener-Pitt Tauberian theorem [**8**, p. 163].

The paper is organized as follows. In §2, we introduce the notation and sketch the extension of the Fourier-Wiener transform to abstract Wiener space. In §3 we introduce the $\mathscr{L}_2$-closure theorem. The results on the $\mathscr{L}_1$-closure theorem and the Tauberian theorem are given in the last section.

2. **Preliminaries and notation.** Let H be a real separable Hilbert-space and suppose $\|\cdot\|_1$ is a measurable norm [**5**, p. 374]. Then it is known [**6**] that $\|\cdot\|_1$ is weaker than $\|\cdot\|$ on H and the canonical Gaussian distribution on H induces a Gaussian measure μ on the Borel subsets of B, the completion of H under $\|\cdot\|_1$. The triple $(B, \mu, \|\cdot\|_1)$ is called an abstract Wiener space [**6**] with generating Hilbert space H.

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If y is in B^* (the topological dual of B) then the restriction of y to H is continuous on H because $\|\cdot\|_1$ is weaker than $\|\cdot\|$ on H. Since H is dense in B, the restriction is a one-to-one linear map of B^* into H^*.

We shall identify B^* with a subset of H^* and H^* with H. Now B^* is dense in $H^*=H$ since B^* separates points of H, and hence B^* is dense in B. Furthermore, since B and H are separable we have a countable set $\{\alpha_n\}$ of B^* such that $\{\alpha_n\}$ is an orthonormal basis of H. Here orthogonality is with respect to inner product in H. For each n, (x, α_n), $x \in B$, will mean, of course, α_n applied to the vector x. Since $\alpha_n \in B^* \subseteq H$ and $\|\alpha_n\|=1$, it follows that $(\cdot, \alpha_n)$ is a bounded linear functional on B and that it has Gaussian distribution with mean zero and variance one with respect to the measure μ on B. If $h \in H$, we define

$$(x, h)^\sim = \lim_n (x, h_n) \quad \text{where } (x, h_n) = \sum_{k=1}^{n} c_k(x, \alpha_k)$$

and $c_k=(h, \alpha_k)$. We note that $\{(x, \alpha_k)\}$ is a sequence of independent Gaussian functions with mean zero and variance one, and $\sum_{k=1}^{\infty} c_k^2<\infty$, since $h \in H$ and $\{\alpha_k\}$ is complete orthonormal set in H. This implies $(x, h)^\sim$ exists for almost all $x \in B$ and it has a Gaussian distribution with mean zero and variance $\|h\|^2$. Furthermore, it is easy to show that $(x, h)^\sim$ equals (x, h) almost everywhere on B if $h \in B^*$, $(x, h)^\sim$ is independent of the complete orthonormal set used in its definition, and finally, if $h_1, h_2, \cdots, h_n$ are orthonormal then $(x, h_1)^\sim, \cdots, (x, h_n)^\sim$ are independent Gaussian functions with mean zero and variance one.

If f is a polynomial in the variables $\{(x, \alpha_k)\}$ then we define the Fourier-Wiener transform $\mathscr{F}$ of f following Cameron and Martin **[2]**, **[3]** and Segal **[9]** by

$$\mathscr{F}f(y) = \int_B f(\sqrt{2}x + iy)\, d\mu(x) \qquad (y \in B). \tag{2.1}$$

Here, if $f(x)=g((x, \alpha_1), \cdots, (x, \alpha_N))$ where g is a function of N complex variables, then $f(u+iv)=g[(u, \alpha_1)+i(v, \alpha_1), \cdots, (u, \alpha_N)+i(v, \alpha_N)]$. Then by **[2**, p. 491–492] or **[9**, p. 121], $\mathscr{F}$ is unitary on the class of all polynomials and

$$\mathscr{F}^2 f(y) = f(-y). \tag{2.2}$$

Hence in view of the Fourier-Hermite expansion **[7**, p. 436] one can extend $\mathscr{F}$ to be unitary on $\mathscr{L}_2(\mu)$ such that $\mathscr{F}^2 f(y)=f(-y)$. We also remark that $\mathscr{F}f$ can be evaluated as in (2.1) for a much larger class of functionals than the polynomials indicated.

3. **Wiener's theorem for $\mathscr{L}_p(\mu)$, $0<p\leqq 2$.** Let λ be the Lebesgue measure on the Borel subsets of the space of real numbers. The classical

theorem of N. Wiener says that any translation invariant (closed) subspace of $L_2(\lambda)$ consists precisely of those functions whose Fourier transforms vanish on a measurable set. Our purpose in this section is to generalize Wiener's theorem to $\mathscr{L}_2(\mu)$. Let $f \in \mathscr{L}_2(\mu)$, then for each $h \in H$, $(U_h f)(y) = f(y+h)\exp\{-\frac{1}{2}(y, h)^{\sim} - \frac{1}{4}\|h\|^2\}$ is called the $\mathscr{L}_2$-translate of f. It is easy to check by the translation theorem as given in [7, p. 435] that, for each $h \in H$, U_h is a unitary operator on $\mathscr{L}_2(\mu)$ onto $\mathscr{L}_2(\mu)$. The main tool of the classical proof is the relation of the Fourier transform of the translate of a function to the Fourier transform of the function itself. The following lemma gives the analogous relation for $f \in \mathscr{L}_2(\mu)$.

LEMMA 3.1. *Let $f \in \mathscr{L}_2(\mu)$, then, for each $h \in H$,*

$$\mathscr{F}(U_h f)(y) = \exp\left\{-\frac{i}{2}(y, h)^{\sim}\right\}\mathscr{F}f(y)$$

with μ-measure one on B.

PROOF. We first assume that f is a polynomial in some of the variables $\{(x, \alpha_k) : k \geqq 1\}$ where $\{\alpha_k\}$ is completely orthonormal in $B^* \subseteq H^*$. Then for $h \in H$ we obtain

$$\begin{aligned}
\mathscr{F}f(y) &= \int_B f(\sqrt{2}x + iy)\, d\mu(x) \\
&= \int_B f\left(\sqrt{2}\left(x + \frac{h}{\sqrt{2}}\right) + iy\right)\exp\left\{-\left(x, \frac{h}{\sqrt{2}}\right)^{\sim} - \frac{(h, h)}{4}\right\} d\mu(x) \\
&= \int_B f(\sqrt{2}x + iy + h)\exp\left\{-\tfrac{1}{2}(\sqrt{2}x + iy, h)^{\sim} - \frac{(h, h)}{4}\right\} d\mu(x) \\
&\quad \cdot \exp\left\{\frac{i}{2}(y, h)\right\} \\
&= \int_B (U_h f)(\sqrt{2}x + iy)\, d\mu(x) \cdot \exp\left\{\frac{i}{2}(y, h)^{\sim}\right\} \\
&= \exp\left\{\frac{i}{2}(y, h)^{\sim}\right\}\mathscr{F}(U_h f)(y).
\end{aligned}$$

Here the translation by $h \in H$ is handled as indicated due to the translation theorem [7, p. 435].

For arbitrary f in $\mathscr{L}_2(\mu)$ we proceed as follows. The operators U_h and $\mathscr{F}$ are unitary operators on $\mathscr{L}_2(\mu)$ such that, for each $h \in H$ and polynomial f in $\{(x, \alpha_k)\}$, we have the equation

$$\mathscr{F}(U_h f) = e_h(\cdot)\mathscr{F}f$$

where $e_h(\cdot) = \exp\{-i(\cdot, h)^{\sim}/2\}$ and the equation is understood as $\mathscr{L}_2$-equivalence. Since such polynomials are dense in $\mathscr{L}_2(\mu)$ [7, p. 436], the proof follows.

The following is our version of Wiener's theorem.

THEOREM 3.2. *Let* $\mathfrak{W}$ *be a translation invariant closed subspace of* $\mathscr{L}_2(\mu)$*; i.e.,* $U_h\mathfrak{W}\subseteq\mathfrak{W}$*, for each* $h\in H$*. Then there exists a measurable subset* E *of* B *such that* $\mathfrak{W}=M_E$ *where* $M_E=\{f\,|\,f\in\mathscr{L}_2(\mu)$ *such that* $\mathscr{F}f(y)=0$ *a.e.* $[\mu]$ *for all* $y\in E\}$*. Conversely, each* M_E *is translation invariant. Further,* $M_A=M_B$ *if and only if* $\mu(A\triangle B)=0$.

PROOF. The converse being obvious from Lemma 3.1 we proceed to the direct part. We note that the proof is basically classical. Let $\mathfrak{W}$ be a translation invariant closed subspace of $\mathscr{L}_2(\mu)$. Let $\mathfrak{N}=\mathscr{F}\mathfrak{W}$. Since $\mathscr{F}$ is unitary, $\mathfrak{N}$ is a closed subspace of $\mathscr{L}_2(\mu)$ and in view of Lemma 3.1, $\mathfrak{N}$ is invariant under multiplication by $e_h(\cdot)$. Let P be the orthogonal projection of $\mathscr{L}_2(\mu)$ onto $\mathfrak{N}$. Then $f-Pf\perp Pg$ for all $f,g\in\mathscr{L}_2(\mu)$ and since $\mathfrak{N}$ is invariant under multiplication by $e_h(\cdot)$, we have, for all $h\in H$,

$$\int_B (f(x)-(Pf)(x))\overline{(Pg)(x)}e_{-h}(x)\,d\mu(x) = 0.$$

Since $B^*\subset H$, B is separable, and every (complex) measure on B is uniquely determined by its Fourier transform, the above equation implies that, for all $f,g\in\mathscr{L}_2(\mu)$,

$$f(x)(Pg)(x) = (Pf)(x)(Pg)(x) \quad \text{a.e. } \mu.$$

Interchanging the roles of f and g we obtain

$$f(x)(Pg)(x) = g(x)(Pf)(x) \quad \text{a.e. } \mu.$$

Taking $g\equiv 1$ we get

$$(Pf)(x) = \varphi(x)f(x) \quad \text{a.e. } \mu \text{ for all } f\in\mathscr{L}_2(\mu),$$

where $\varphi(x)$ is the projection of the function identically one onto $\mathfrak{N}$. But $P^2=P$ implies $\varphi^2=\varphi$ a.e. μ. Hence $\varphi(x)=0$ or 1 a.e. μ and we let $E=\{x;\,\varphi(x)=0\}$. Since $f\in\mathfrak{N}$ iff $f=Pf=\varphi\cdot f$ we see that $\mathfrak{N}$ consists of those functions which vanish at least on E, giving $\mathfrak{W}=M_E$. The uniqueness part being simple is omitted.

COROLLARY 3.1 (WIENER [**10**, p. 267]). *Let* $f\in\mathscr{L}_2(\mu)$ *such that* $\mu\{y;\,(\mathscr{F}f)(y)=0\}=0$. *Then the linear manifold generated by the* $\mathscr{L}_2$*-translates of* f *is dense in* $\mathscr{L}_p(\mu)$, $0<p\leqq 2$.

PROOF. Since $\mathscr{L}_2(\mu)$ is dense in $\mathscr{L}_p(\mu)$ is suffices to show the theorem in the case $p=2$. Let $\mathfrak{W}$ be the closed subspace of $\mathscr{L}_2(\mu)$ generated by the $\mathscr{L}_2$-translates of f. Then clearly $\mathfrak{W}$ is translation invariant and since $\mathscr{F}f(y)\neq 0$ with μ-measure one $\mathfrak{W}=M_\varnothing$ where $\varnothing$ is the empty set. By definition $M_\varnothing=\mathscr{L}_2(\mu)$ so the proof is complete.

4. **Translation in $\mathscr{L}_1(\mu)$ and a Tauberian theorem.** The translation operation $U_h f$ ($h \in H$) as used previously is an isometry on $\mathscr{L}_2(\mu)$ onto $\mathscr{L}_2(\mu)$, but it is not an isometry in $\mathscr{L}_1(\mu)$ unless one has the trivial situation $h=0$. The translation for functions in $\mathscr{L}_1(\mu)$ which is an isometry is the following:

$$W_h f(x) = f(x+h)\exp\{-(x, h)^\sim - \tfrac{1}{2}(h, h)\} \qquad (h \in H). \tag{4.1}$$

That it is an isometry from $\mathscr{L}_1(\mu)$ onto $\mathscr{L}_1(\mu)$ follows easily from the translation theorem of Cameron-Martin for this setting [7, p. 435]. As we shall see later it also behaves nicely with respect to convolution.

Now the Fourier-Wiener transform of the translate $W_h f$ (assuming f is in $\mathscr{L}_2(\mu)$ so that $\mathscr{F}f$ is defined) does not retain the crucial property of Lemma 3.1. However, if we use a slightly modified Fourier-Wiener transform we can obtain a similar result.

We now define the $\mathscr{L}_1$-Fourier-Wiener transform of a function $f \in \mathscr{L}_1(\mu)$ by

$$\mathscr{F}_1 f(h) = \int_B \exp\{i(x, h)^\sim\} f(x)\mu(dx) \quad \text{for } h \in H. \tag{4.2}$$

In view of the translation theorem [7, p. 435] we have, for each $h_0 \in H$,

$$\mathscr{F}_1(W_{h_0} f)(h) = \exp\{-i(h_0, h)\}\mathscr{F}_1 f(h) \quad \text{for } h \in H. \tag{4.3}$$

We are now ready to prove a theorem for the $\mathscr{L}_1$-transform and $\mathscr{L}_1$-translates. As one might guess we make use of Wiener's original theorem in some way. We also point out that the $\mathscr{L}_1$-transform in (4.2) is closely related to the Fourier-Wiener transform of Cameron and Martin in [2] which was subsequently modified in [3] becoming the $\mathscr{L}_2$-transform.

A function f is a tame function on B if there exist vectors h_1 $h_2, \cdots, h_k \in H$ such that

$$f(x) = \Gamma((x, h_1)^\sim, \cdots, (x, h_k)^\sim)$$

where $\Gamma(u_1, u_2, \cdots, u_k)$ is a Borel measurable function on R_k. By orthogonalization of $h_1, h_2, \cdots, h_k$ we can always write a tame function in the form

$$f(x) = \Psi((x, \varphi_1)^\sim, \cdots, (x, \varphi_N)^\sim) \tag{4.4}$$

where $\varphi_1, \cdots, \varphi_N$ are orthonormal in H and Ψ is Borel measurable on R_N. Hence we lose no generality in assuming (4.4) is the case.

THEOREM 4.1. *Let $f \in \mathscr{L}_1(\mu)$ be of the form* (4.4) *and $\mathscr{F}_1 f(h) \neq 0$ for all $h \in H$. Then $\mathfrak{W}_f$, the linear manifold generated by the $\mathscr{L}_1$-translates of f, is dense in $\mathscr{L}_p(\mu)$ for $0 \leqq p \leqq 1$.*

PROOF. Let $\{\varphi_j : j \geqq N+1\}$ be an orthonormal set in H such that $\{\varphi_j : j \geqq 1\}$ is complete. Then, using the Fourier-Hermite expansion of functions in $\mathscr{L}_2(\mu)$ due to Cameron and Martin [4] and appearing in this generality in Lemma 2.2 of [7], we see that tame functions of the form

$$L(x) = \Phi[(x, \varphi_1)^{\sim}, \cdots, (x, \varphi_m)^{\sim}] \qquad (m = 1, 2, \cdots) \tag{4.5}$$

where $L \in \mathscr{L}_2(\mu)$ are dense in $\mathscr{L}_2(\mu)$. Since each element in $\mathscr{L}_1(\mu)$ can be approximated in $\mathscr{L}_1$-norm by a function in $\mathscr{L}_2(\mu)$ and the $\mathscr{L}_2(\mu)$-norm is greater than the $\mathscr{L}_1$-norm on $\mathscr{L}_2(\mu)$ it follows that functions of the form (4.5) are dense in $\mathscr{L}_1(\mu)$ with respect to the $\mathscr{L}_1$-norm. Hence the theorem is proved if the $\mathscr{L}_1$-translates approximate any function $L \in \mathscr{L}_2(\mu)$ which is of the form (4.5).

Now any tame function of the form (4.5) with $m < N$ can be written as a tame function with $m = N$ by simply multiplying $\Phi(u_1, \cdots, u_m)$ by $\Phi_1(u_{n+1}, \cdots, u_N) \equiv 1$ and hence we can assume $m \geqq N$. On the other hand, if $m > N$ we then write

$$f(x) = \Psi[(x, \varphi_1)^{\sim}, \cdots, (x, \varphi_N)^{\sim}] \cdot \Phi_2[(x, \varphi_{N+1})^{\sim}, \cdots, (x, \varphi_m)^{\sim}]$$

where $\Phi_2(u_{N+1}, \cdots, u_m) \equiv 1$. Hence we can assume without loss of generality that $m = N$.

By assumption we have for h of the form $\sum_{i=1}^N a_i \varphi_i$, $(a_1, a_2, \cdots, a_N) \in R_N$,

$$\mathscr{F}_1 f(h) = \int_H \exp\{i(x, h)^{\sim}\} f(x) \mu(dx) \neq 0.$$

Hence we have

$$\begin{aligned} \mathscr{F}_1 f(h) = (2\pi)^{-N/2} \int_{R^N} \exp\left\{ i \sum_{i=1}^N a_i u_i \right\} \Psi[(u_1, \cdots, u_N)] \\ \times \exp\left\{ -\frac{1}{2} \sum_{i=1}^N u_i^2 \right\} du_1, \cdots, du_N \neq 0. \end{aligned} \tag{4.6}$$

Thus the ordinary Fourier transform of

$$\Lambda(v) = (2\pi)^{-N/2} \Psi(v) \exp\{-\tfrac{1}{2} v \cdot v\} \qquad (v \in R_N) \tag{4.7}$$

never vanishes on R_N. Here we use $v \cdot u$ to denote $\sum_1^N u_j v_j$ if $u, v \in R_N$. Thus the ordinary translates of (4.7) generate a dense subset of $L_{1,N}$ where $L_{1,N}$ denotes the integrable Borel functions with respect to Lebesgue measure on R_N. Let $L_{1,N}^g$ denote the Borel functions on R_N which are integrable with respect to the Gaussian density

$$g(v) = (2\pi)^{-N/2} \exp\{-\tfrac{1}{2} v \cdot v\} \qquad (v \in R_N). \tag{4.8}$$

Take an arbitrary tame function $L \in \mathscr{L}_2(\mu)$ of the form (4.5) with $m = N$.

Then $\Phi(v)\in L^g_{1,N}$ and $\Phi(v)g(v)\in L_{1,N}$. Take $\varepsilon>0$. Then by Wiener's theorem for $L_{1,N}$ [**8**, p. 162] there exist translates $t_1, \cdots, t_k\in R_N$ and constants $c_1, \cdots, c_k$ such that

$$\int_{R_N}\left|\sum_{j=1}^{K} c_j\Lambda(v+t_j)-\Phi(v)g(v)\right| dv<\varepsilon. \tag{4.9}$$

Using (4.7), (4.8), and (4.9) we see that

$$\int_{R_N}\left|\sum_{j=1}^{K} c_j\Psi(v+t_j)\exp\{-v\cdot t_j-\tfrac{1}{2}t_j\cdot t_j\}-\Phi(v)\right| g(v)\,dv<\varepsilon. \tag{4.10}$$

Choosing $h_1, \cdots, h_k$ in the subspace of H generated by $\{\varphi_1, \cdots, \varphi_N\}$ and such that $t_j=[(h_j, \varphi_1), \cdots, (h_j, \varphi_N)]$ $(j=1, \cdots, K)$, (4.10) then implies

$$\int_{B}\left|\sum_{j=1}^{K} c_j W_{h_j}f(x)-L(x)\right| d\mu(x)<\varepsilon. \tag{4.11}$$

Hence the tame functions in $\mathscr{L}_2(\mu)$ of the form (4.5) with $m=N$ can be approximated in $\mathscr{L}_1$-norm by our $\mathscr{L}_1$-translates of f. Since $m=N$ represents the general case $\mathfrak{W}_f$ is dense in $\mathscr{L}_1(\mu)$. The proof is now complete since $\mathscr{L}_1(\mu)$ is dense in $\mathscr{L}_p(\mu)$ and the $\mathscr{L}_1$-norm dominates the $\mathscr{L}_p$-distance, $0<p\leqq 1$.

The function $f\in\mathscr{L}_1(\mu)$ is said to be *splittable* with respect to the complete orthonormal set $\{\varphi_k\}$ if there exists a sequence of integers $N_1<N_2<\cdots$ such that, for each integer k,

$$f(x)=L_k(x)\cdot\Gamma_k(x) \quad \text{a.e. } [\mu] \tag{4.12}$$

where $L_k(x)=\Phi_k[(x,, \varphi_1)^\sim, \cdots, (x, \varphi_{N_k})^\sim]$ and Γ_k is $\mathscr{B}_k$ measurable on B where $\mathscr{B}_k$ is the minimal σ-algebra generated by the functionals $\{(\cdot, \varphi_j)^\sim, j\geqq N_k+1\}$.

REMARK 4.1. Since Γ_k is $\mathscr{B}_k$-measurable on B it follows [**1**, p. 395] that there exists a Borel measurable function defined on the space of all real sequences such that

$$\Gamma_k(x)=F((x, \varphi_{N_K+1})^\sim, \cdots) \quad \text{a.e. } [\mu].$$

Also from (4.12) and the fact that $f\in\mathscr{L}_1(\mu)$ we get that L_k, Γ_k are in $\mathscr{L}_1(\mu)$ provided $f\not\equiv 0$.

We say that $f\in\mathscr{L}_1(\mu)$ is *negligibly split* if f is splittable and for every $\varepsilon>0$, there exists a k such that

$$\int_B |\Gamma_k(x)-1|\,d\mu(x)<\varepsilon.$$

It is easy to see that f is then the product of tame functions.

THEOREM 4.2. *Let $f \in \mathscr{L}_1(\mu)$ be negligibly split with respect to the complete orthonormal bases $\{\varphi_k\}$ in H and assume that $(\mathscr{F}_1 f)(h) \neq 0$ for all $h \in H$. Then the linear manifold $\mathfrak{W}_f$ generated by $\mathscr{L}_1$-translates of f is dense in $\mathscr{L}_p(\mu)$, $0 < p \leqq 1$.*

PROOF. In view of the argument given in the proof of Theorem 4.1 it suffices to prove that one can approximate in $\mathscr{L}_1$-norms functions $L \in \mathscr{L}_2(\mu)$ of the form (4.5). Take $\varepsilon > 0$, and suppose $L(x)$ is given. Then there exists N_k such that $N_k \geqq m$ and

$$\int_B |\Gamma_{N_k}(x) - 1| \, d\mu(x) < \varepsilon. \tag{4.13}$$

Again arguing as in Theorem 4.1 we can now assume that $m = N_k$. Also we have

$$\mathscr{F}_1 f(y) = \mathscr{F}_1 L_k(y) \mathscr{F}_1 \Gamma_k(y) e^{(y,y)/2}, \qquad y \in H. \tag{4.14}$$

The above equation holds as indicated since the functionals L_k and Γ_k are independent (probabilistic sense). Since $\mathscr{F}_1 f(y) \neq 0$ we get $\mathscr{F}_1 L_k(y) \neq 0$ by (4.14). Now by the proof of Theorem 4.1 there exist constants $c_1, c_2, \cdots, c_r$ and vectors $h_1, h_2, \cdots, h_r$ in the subspace of H generated by $\{\varphi_1, \cdots, \varphi_{N_k}\}$ such that

$$\int_B \left| \sum_{j=1}^r c_j W_{h_j} L_k(x) - L(x) \right| d\mu(x) < \varepsilon. \tag{4.15}$$

In view of Remark 4.1 it follows that

$$\Gamma_k(x + h_j) = \Gamma_k(x) \qquad (j = 1, \cdots, r)$$

and hence

$$W_{h_j} f(x) = [W_{h_j} L(x)] \cdot \Gamma_k(x) \qquad (j = 1, \cdots, r).$$

Now Γ_k and L_k are independent (probabilistic sense) thus (4.13) and (4.15) imply

$$\begin{aligned} \int_B \left| \sum_{j=1}^r c_j W_{h_j} f(x) - \sum_{j=1}^r c_j W_{h_j} L_k(x) \right| d\mu(x) \\ = \int_B \left| \sum_{j=1}^r c_j W_{h_j} L_k(x) [\Gamma_k(x) - 1] \right| d\mu(x) \\ = \int_B \left| \sum_{j=1}^r c_j W_{h_j} L_k(x) \right| d\mu(x) \cdot \int_B |\Gamma_k(x) - 1| \, d\mu(x) \\ \leqq \left[\int_B |L(x)| \, d\mu(x) + \varepsilon \right] \times \varepsilon. \end{aligned} \tag{4.16}$$

Combining (4.15) and (4.16) along with $\varepsilon>0$ being arbitrary completes the proof.

REMARK 4.2. If f is a tame function of the form (4.4) then f is easily seen to be negligibly split with respect to the complete orthonormal basis $\{\varphi_k\}$ where $\{\varphi_1, \cdots, \varphi_N\}$ are as in (4.4). Thus Theorem 4.2 actually implies Theorem 4.1, but we proved both theorems since a direct proof of Theorem 4.2 would involve about the same amount of effort.

For an example of a function $f\in\mathscr{L}_1(\mu)$ which satisfies Theorem 4.2 but not Theorem 4.1 consider

$$f(x) = \exp\left\{\sum_{k=1}^{\infty} \lambda_k (x, \varphi_k)^{\sim 2}\right\}$$

where $\{\lambda_k\}$ is a sequence of positive numbers such that $\sum_{k=1}^{\infty} \lambda_k < \frac{1}{2}$ and $\{\varphi_k\}$ is an orthonormal set in H. Then $f\in\mathscr{L}_1(\mu)$,

$$\mathscr{F}_1 f(h) = \exp\{-\sum_{k=1}^{\infty} (h, \varphi_k)^2/2(1 - 2\lambda_k) \neq 0$$

and f is negligibly split with respect to $\{\varphi_k\}$.

If f and φ are measurable functions on B we define the *convolution of f and φ* by the usual formula

$$f * \varphi(x) = \int_B f(y)\varphi(y - x)\, d\mu(y). \tag{4.17}$$

In this setup convolution is not always commutative since the measure μ is not translation invariant. It does, however, act in a normal way with respect to translation if we use $\mathscr{L}_1$-translates and we show this in the next lemma.

LEMMA 4.3. *Suppose $f\in\mathscr{L}_1(\mu)$ and $\varphi\in\mathscr{L}_\infty(\mu)$. Then*
(1) *$f * \varphi(x)$ exists for each x in B,*
(2) *$(W_h f) * \varphi(x) = f * \varphi(x+h)$ for each $x\in B$ and $h\in H$.*

PROOF. Since φ is in $\mathscr{L}_\infty(\mu)$ and $\varphi(y-x)$ is measurable as a function of y for each $x\in B$ the conclusion of (1) is immediate. Now $f\in\mathscr{L}_1(\mu)$ iff $W_h f\in\mathscr{L}_1(\mu)$ for each $h\in H$ [7, p. 435] so it follows that $W_h f * \varphi(x)$ exists on B for each $x\in H$. Further,

$$\begin{aligned} W_h f * \varphi(x) &= \int_B W_h f(y)\varphi(y - x)\, d\mu(y) \\ &= \int_B f(y + h)\exp\{-(x, h)^{\sim} - \tfrac{1}{2}(h, h)^{\sim}\}\varphi(y - x)\, d\mu(y) \\ &= \int_B f(y)\varphi(y - x - h)\, d\mu(y) = f * \varphi(x + h) \end{aligned}$$

where the third equality follows from the translation theorem [7, p. 435]. Hence (2) holds.

We now state a Tauberian theorem for abstract Wiener spaces. Its proof is exactly as in [**10**, p. 285] if one uses the definition of convolution in (4.17) and the $\mathscr{L}_1$-translates. If φ is defined on B we say $\lim_{x\to\infty} \varphi(x)=c$ if for each $\varepsilon>0$ there exists a bounded set E such that $|\varphi(x)-c|<\varepsilon$ on $B-E$.

THEOREM 4.3. *Suppose* $f\in\mathscr{L}_1(\mu)$ *and* $\mathscr{F}_1 f(y)\neq 0$, $y\in H$. *Further, assume* $\varphi\in\mathscr{L}_\infty(\mu)$ *and c is a constant such that*

$$\lim_{x\to\infty} f * \varphi(x) = c\mathscr{F}_1 f(0).$$

Then, if f is negligibly split, we have $\lim g*\varphi(x)=c\int_B g(x)\,d\mu(x)$ *for all* $g\in\mathscr{L}_1(\mu)$.

As a final remark we mention that Pitt's Tauberian theorem, as it appears in [**8**, p. 163], can also be proved in this setting. Here, however, we would define slowly oscillating in terms of a norm bounded set and a norm bounded neighborhood of zero. Since open subsets of B have positive μ-measure the proof is as in [**8**].

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