

THE WIENER CLOSURE THEOREMS FOR ABSTRACT WIENER SPACES¹

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ABSTRACT. We introduce \mathcal{L}_1 and \mathcal{L}_2 translates for functions in $\mathcal{L}_1(\mu)$ and $\mathcal{L}_2(\mu)$ where μ is a Gaussian measure on a Banach space. With these translates and the Fourier-Wiener transforms defined by Cameron and Martin we obtain Wiener's closure theorem in $\mathcal{L}_2(\mu)$ and in $\mathcal{L}_1(\mu)$. Using the $\mathcal{L}_1(\mu)$ results we indicate the analogue of the Wiener-Pitt Tauberian theorems for this setup.

1. Introduction. Let μ be the Wiener measure on $C[0, 1]$ and $\mathcal{L}_2(\mu)$ be the space of square integrable Borel functions with respect to μ . For $f \in \mathcal{L}_2(\mu)$, the Fourier-Wiener transform was defined by Cameron and Martin [3]. In this paper we extend this notion to abstract Wiener spaces [6] and obtain an analogue of Wiener's closure theorem [10] for $\mathcal{L}_2(\mu)$. Our main effort however is to obtain an analogue of Wiener's closure theorem for $\mathcal{L}_1(\mu)$. From this theorem one can easily derive the Wiener-Pitt Tauberian theorem [8, p. 163].

The paper is organized as follows. In §2, we introduce the notation and sketch the extension of the Fourier-Wiener transform to abstract Wiener space. In §3 we introduce the \mathcal{L}_2 -closure theorem. The results on the \mathcal{L}_1 -closure theorem and the Tauberian theorem are given in the last section.

2. Preliminaries and notation. Let H be a real separable Hilbert-space and suppose $\|\cdot\|_1$ is a measurable norm [5, p. 374]. Then it is known [6] that $\|\cdot\|_1$ is weaker than $\|\cdot\|$ on H and the canonical Gaussian distribution on H induces a Gaussian measure μ on the Borel subsets of B , the completion of H under $\|\cdot\|_1$. The triple $(B, \mu, \|\cdot\|_1)$ is called an abstract Wiener space [6] with generating Hilbert space H .

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If y is in B^* (the topological dual of B) then the restriction of y to H is continuous on H because $\|\cdot\|_1$ is weaker than $\|\cdot\|$ on H . Since H is dense in B , the restriction is a one-to-one linear map of B^* into H^* .

We shall identify B^* with a subset of H^* and H^* with H . Now B^* is dense in $H^*=H$ since B^* separates points of H , and hence B^* is dense in B . Furthermore, since B and H are separable we have a countable set $\{\alpha_n\}$ of B^* such that $\{\alpha_n\}$ is an orthonormal basis of H . Here orthogonality is with respect to inner product in H . For each n , (x, α_n) , $x \in B$, will mean, of course, α_n applied to the vector x . Since $\alpha_n \in B^* \subseteq H$ and $\|\alpha_n\|=1$, it follows that (\cdot, α_n) is a bounded linear functional on B and that it has Gaussian distribution with mean zero and variance one with respect to the measure μ on B . If $h \in H$, we define

$$(x, h) \sim = \lim_n (x, h_n) \quad \text{where } (x, h_n) = \sum_{k=1}^n c_k(x, \alpha_k)$$

and $c_k = (h, \alpha_k)$. We note that $\{(x, \alpha_k)\}$ is a sequence of independent Gaussian functions with mean zero and variance one, and $\sum_{k=1}^\infty c_k^2 < \infty$, since $h \in H$ and $\{\alpha_k\}$ is complete orthonormal set in H . This implies $(x, h) \sim$ exists for almost all $x \in B$ and it has a Gaussian distribution with mean zero and variance $\|h\|^2$. Furthermore, it is easy to show that $(x, h) \sim$ equals (x, h) almost everywhere on B if $h \in B^*$, $(x, h) \sim$ is independent of the complete orthonormal set used in its definition, and finally, if h_1, h_2, \dots, h_n are orthonormal then $(x, h_1) \sim, \dots, (x, h_n) \sim$ are independent Gaussian functions with mean zero and variance one.

If f is a polynomial in the variables $\{(x, \alpha_k)\}$ then we define the Fourier-Wiener transform \mathcal{F} of f following Cameron and Martin [2], [3] and Segal [9] by

$$(2.1) \quad \mathcal{F}f(y) = \int_B f(\sqrt{2}x + iy) d\mu(x) \quad (y \in B).$$

Here, if $f(x) = g((x, \alpha_1), \dots, (x, \alpha_N))$ where g is a function of N complex variables, then $f(u+iv) = g[(u, \alpha_1) + i(v, \alpha_1), \dots, (u, \alpha_N) + i(v, \alpha_N)]$. Then by [2, p. 491-492] or [9, p. 121], \mathcal{F} is unitary on the class of all polynomials and

$$(2.2) \quad \mathcal{F}^2 f(y) = f(-y).$$

Hence in view of the Fourier-Hermite expansion [7, p. 436] one can extend \mathcal{F} to be unitary on $\mathcal{L}_2(\mu)$ such that $\mathcal{F}^2 f(y) = f(-y)$. We also remark that $\mathcal{F}f$ can be evaluated as in (2.1) for a much larger class of functionals than the polynomials indicated.

3. **Wiener's theorem for $\mathcal{L}_p(\mu)$, $0 < p \leq 2$.** Let λ be the Lebesgue measure on the Borel subsets of the space of real numbers. The classical

theorem of N. Wiener says that any translation invariant (closed) subspace of $L_2(\lambda)$ consists precisely of those functions whose Fourier transforms vanish on a measurable set. Our purpose in this section is to generalize Wiener's theorem to $\mathcal{L}_2(\mu)$. Let $f \in \mathcal{L}_2(\mu)$, then for each $h \in H$, $(U_h f)(y) = f(y+h) \exp\{-\frac{1}{2}(y, h) \sim -\frac{1}{4} \|h\|^2\}$ is called the \mathcal{L}_2 -translate of f . It is easy to check by the translation theorem as given in [7, p. 435] that, for each $h \in H$, U_h is a unitary operator on $\mathcal{L}_2(\mu)$ onto $\mathcal{L}_2(\mu)$. The main tool of the classical proof is the relation of the Fourier transform of the translate of a function to the Fourier transform of the function itself. The following lemma gives the analogous relation for $f \in \mathcal{L}_2(\mu)$.

LEMMA 3.1. *Let $f \in \mathcal{L}_2(\mu)$, then, for each $h \in H$,*

$$\mathcal{F}(U_h f)(y) = \exp\left\{-\frac{i}{2}(y, h) \sim\right\} \mathcal{F}f(y)$$

with μ -measure one on B .

PROOF. We first assume that f is a polynomial in some of the variables $\{(x, \alpha_k): k \geq 1\}$ where $\{\alpha_k\}$ is completely orthonormal in $B^* \subseteq H^*$. Then for $h \in H$ we obtain

$$\begin{aligned} \mathcal{F}f(y) &= \int_B f(\sqrt{2}x + iy) d\mu(x) \\ &= \int_B f\left(\sqrt{2}\left(x + \frac{h}{\sqrt{2}}\right) + iy\right) \exp\left\{-\left(x, \frac{h}{\sqrt{2}}\right) \sim -\frac{(h, h)}{4}\right\} d\mu(x) \\ &= \int_B f(\sqrt{2}x + iy + h) \exp\left\{-\frac{1}{2}(\sqrt{2}x + iy, h) \sim -\frac{(h, h)}{4}\right\} d\mu(x) \\ &\quad \cdot \exp\left\{\frac{i}{2}(y, h)\right\} \\ &= \int_B (U_h f)(\sqrt{2}x + iy) d\mu(x) \cdot \exp\left\{\frac{i}{2}(y, h) \sim\right\} \\ &= \exp\left\{\frac{i}{2}(y, h) \sim\right\} \mathcal{F}(U_h f)(y). \end{aligned}$$

Here the translation by $h \in H$ is handled as indicated due to the translation theorem [7, p. 435].

For arbitrary f in $\mathcal{L}_2(\mu)$ we proceed as follows. The operators U_h and \mathcal{F} are unitary operators on $\mathcal{L}_2(\mu)$ such that, for each $h \in H$ and polynomial f in $\{(x, \alpha_k)\}$, we have the equation

$$\mathcal{F}(U_h f) = e_h(\cdot) \mathcal{F}f$$

where $e_h(\cdot) = \exp\{-i(\cdot, h) \sim / 2\}$ and the equation is understood as \mathcal{L}_2 -equivalence. Since such polynomials are dense in $\mathcal{L}_2(\mu)$ [7, p. 436], the proof follows.

The following is our version of Wiener's theorem.

THEOREM 3.2. *Let \mathfrak{W} be a translation invariant closed subspace of $\mathcal{L}_2(\mu)$; i.e., $U_h\mathfrak{W} \subseteq \mathfrak{W}$, for each $h \in H$. Then there exists a measurable subset E of B such that $\mathfrak{W} = M_E$ where $M_E = \{f \mid f \in \mathcal{L}_2(\mu) \text{ such that } \mathcal{F}f(y) = 0 \text{ a.e. } [\mu] \text{ for all } y \in E\}$. Conversely, each M_E is translation invariant. Further, $M_A = M_B$ if and only if $\mu(A \Delta B) = 0$.*

PROOF. The converse being obvious from Lemma 3.1 we proceed to the direct part. We note that the proof is basically classical. Let \mathfrak{W} be a translation invariant closed subspace of $\mathcal{L}_2(\mu)$. Let $\mathfrak{N} = \mathcal{F}\mathfrak{W}$. Since \mathcal{F} is unitary, \mathfrak{N} is a closed subspace of $\mathcal{L}_2(\mu)$ and in view of Lemma 3.1, \mathfrak{N} is invariant under multiplication by $e_h(\cdot)$. Let P be the orthogonal projection of $\mathcal{L}_2(\mu)$ onto \mathfrak{N} . Then $f - Pf \perp Pg$ for all $f, g \in \mathcal{L}_2(\mu)$ and since \mathfrak{N} is invariant under multiplication by $e_h(\cdot)$, we have, for all $h \in H$,

$$\int_B (f(x) - (Pf)(x)) \overline{(Pg)(x)} e_{-h}(x) d\mu(x) = 0.$$

Since $B^* \subset H$, B is separable, and every (complex) measure on B is uniquely determined by its Fourier transform, the above equation implies that, for all $f, g \in \mathcal{L}_2(\mu)$,

$$f(x)(Pg)(x) = (Pf)(x)(Pg)(x) \text{ a.e. } \mu.$$

Interchanging the roles of f and g we obtain

$$f(x)(Pg)(x) = g(x)(Pf)(x) \text{ a.e. } \mu.$$

Taking $g \equiv 1$ we get

$$(Pf)(x) = \varphi(x)f(x) \text{ a.e. } \mu \text{ for all } f \in \mathcal{L}_2(\mu),$$

where $\varphi(x)$ is the projection of the function identically one onto \mathfrak{N} . But $P^2 = P$ implies $\varphi^2 = \varphi$ a.e. μ . Hence $\varphi(x) = 0$ or 1 a.e. μ and we let $E = \{x; \varphi(x) = 0\}$. Since $f \in \mathfrak{N}$ iff $f = Pf = \varphi \cdot f$ we see that \mathfrak{N} consists of those functions which vanish at least on E , giving $\mathfrak{W} = M_E$. The uniqueness part being simple is omitted.

COROLLARY 3.1 (WIENER [10, p. 267]). *Let $f \in \mathcal{L}_2(\mu)$ such that $\mu\{y; (\mathcal{F}f)(y) = 0\} = 0$. Then the linear manifold generated by the \mathcal{L}_2 -translates of f is dense in $\mathcal{L}_p(\mu)$, $0 < p \leq 2$.*

PROOF. Since $\mathcal{L}_2(\mu)$ is dense in $\mathcal{L}_p(\mu)$ it suffices to show the theorem in the case $p = 2$. Let \mathfrak{W} be the closed subspace of $\mathcal{L}_2(\mu)$ generated by the \mathcal{L}_2 -translates of f . Then clearly \mathfrak{W} is translation invariant and since $\mathcal{F}f(y) \neq 0$ with μ -measure one $\mathfrak{W} = M_\emptyset$ where \emptyset is the empty set. By definition $M_\emptyset = \mathcal{L}_2(\mu)$ so the proof is complete.

4. Translation in $\mathcal{L}_1(\mu)$ and a Tauberian theorem. The translation operation $U_h f$ ($h \in H$) as used previously is an isometry on $\mathcal{L}_2(\mu)$ onto $\mathcal{L}_2(\mu)$, but it is not an isometry in $\mathcal{L}_1(\mu)$ unless one has the trivial situation $h=0$. The translation for functions in $\mathcal{L}_1(\mu)$ which is an isometry is the following:

$$(4.1) \quad W_h f(x) = f(x + h) \exp\{-(x, h)^\sim - \frac{1}{2}(h, h)\} \quad (h \in H).$$

That it is an isometry from $\mathcal{L}_1(\mu)$ onto $\mathcal{L}_1(\mu)$ follows easily from the translation theorem of Cameron-Martin for this setting [7, p. 435]. As we shall see later it also behaves nicely with respect to convolution.

Now the Fourier-Wiener transform of the translate $W_h f$ (assuming f is in $\mathcal{L}_2(\mu)$ so that $\mathcal{F}f$ is defined) does not retain the crucial property of Lemma 3.1. However, if we use a slightly modified Fourier-Wiener transform we can obtain a similar result.

We now define the \mathcal{L}_1 -Fourier-Wiener transform of a function $f \in \mathcal{L}_1(\mu)$ by

$$(4.2) \quad \mathcal{F}_1 f(h) = \int_B \exp\{i(x, h)^\sim\} f(x) \mu(dx) \quad \text{for } h \in H.$$

In view of the translation theorem [7, p. 435] we have, for each $h_0 \in H$,

$$(4.3) \quad \mathcal{F}_1(W_{h_0} f)(h) = \exp\{-i(h_0, h)\} \mathcal{F}_1 f(h) \quad \text{for } h \in H.$$

We are now ready to prove a theorem for the \mathcal{L}_1 -transform and \mathcal{L}_1 -translates. As one might guess we make use of Wiener's original theorem in some way. We also point out that the \mathcal{L}_1 -transform in (4.2) is closely related to the Fourier-Wiener transform of Cameron and Martin in [2] which was subsequently modified in [3] becoming the \mathcal{L}_2 -transform.

A function f is a tame function on B if there exist vectors $h_1, h_2, \dots, h_k \in H$ such that

$$f(x) = \Gamma((x, h_1)^\sim, \dots, (x, h_k)^\sim)$$

where $\Gamma(u_1, u_2, \dots, u_k)$ is a Borel measurable function on R_k . By orthogonalization of h_1, h_2, \dots, h_k we can always write a tame function in the form

$$(4.4) \quad f(x) = \Psi((x, \varphi_1)^\sim, \dots, (x, \varphi_N)^\sim)$$

where $\varphi_1, \dots, \varphi_N$ are orthonormal in H and Ψ is Borel measurable on R_N . Hence we lose no generality in assuming (4.4) is the case.

THEOREM 4.1. *Let $f \in \mathcal{L}_1(\mu)$ be of the form (4.4) and $\mathcal{F}_1 f(h) \neq 0$ for all $h \in H$. Then \mathfrak{W}_f , the linear manifold generated by the \mathcal{L}_1 -translates of f , is dense in $\mathcal{L}_p(\mu)$ for $0 \leq p \leq 1$.*

PROOF. Let $\{\varphi_j; j \geq N+1\}$ be an orthonormal set in H such that $\{\varphi_j; j \geq 1\}$ is complete. Then, using the Fourier-Hermite expansion of functions in $\mathcal{L}_2(\mu)$ due to Cameron and Martin [4] and appearing in this generality in Lemma 2.2 of [7], we see that tame functions of the form

$$(4.5) \quad L(x) = \Phi[(x, \varphi_1)^\sim, \dots, (x, \varphi_m)^\sim] \quad (m = 1, 2, \dots)$$

where $L \in \mathcal{L}_2(\mu)$ are dense in $\mathcal{L}_2(\mu)$. Since each element in $\mathcal{L}_1(\mu)$ can be approximated in \mathcal{L}_1 -norm by a function in $\mathcal{L}_2(\mu)$ and the $\mathcal{L}_2(\mu)$ -norm is greater than the \mathcal{L}_1 -norm on $\mathcal{L}_2(\mu)$ it follows that functions of the form (4.5) are dense in $\mathcal{L}_1(\mu)$ with respect to the \mathcal{L}_1 -norm. Hence the theorem is proved if the \mathcal{L}_1 -translates approximate any function $L \in \mathcal{L}_2(\mu)$ which is of the form (4.5).

Now any tame function of the form (4.5) with $m < N$ can be written as a tame function with $m = N$ by simply multiplying $\Phi(u_1, \dots, u_m)$ by $\Phi_1(u_{m+1}, \dots, u_N) \equiv 1$ and hence we can assume $m \geq N$. On the other hand, if $m > N$ we then write

$$f(x) = \Psi[(x, \varphi_1)^\sim, \dots, (x, \varphi_N)^\sim] \cdot \Phi_2[(x, \varphi_{N+1})^\sim, \dots, (x, \varphi_m)^\sim]$$

where $\Phi_2(u_{N+1}, \dots, u_m) \equiv 1$. Hence we can assume without loss of generality that $m = N$.

By assumption we have for h of the form $\sum_{i=1}^N a_i \varphi_i$, $(a_1, a_2, \dots, a_N) \in R_N$,

$$\mathcal{F}_1 f(h) = \int_H \exp\{i(x, h)^\sim\} f(x) \mu(dx) \neq 0.$$

Hence we have

$$(4.6) \quad \begin{aligned} \mathcal{F}_1 f(h) &= (2\pi)^{-N/2} \int_{R^N} \exp\left\{i \sum_{i=1}^N a_i u_i\right\} \Psi[(u_1, \dots, u_N)] \\ &\quad \times \exp\left\{-\frac{1}{2} \sum_{i=1}^N u_i^2\right\} du_1, \dots, du_N \neq 0. \end{aligned}$$

Thus the ordinary Fourier transform of

$$(4.7) \quad \Lambda(v) = (2\pi)^{-N/2} \Psi(v) \exp\{-\frac{1}{2} v \cdot v\} \quad (v \in R_N)$$

never vanishes on R_N . Here we use $v \cdot u$ to denote $\sum_{i=1}^N u_i v_i$ if $u, v \in R_N$. Thus the ordinary translates of (4.7) generate a dense subset of $L_{1,N}$ where $L_{1,N}$ denotes the integrable Borel functions with respect to Lebesgue measure on R_N . Let $L_{1,N}^g$ denote the Borel functions on R_N which are integrable with respect to the Gaussian density

$$(4.8) \quad g(v) = (2\pi)^{-N/2} \exp\{-\frac{1}{2} v \cdot v\} \quad (v \in R_N).$$

Take an arbitrary tame function $L \in \mathcal{L}_2(\mu)$ of the form (4.5) with $m = N$.

Then $\Phi(v) \in L_{1,N}^q$ and $\Phi(v)g(v) \in L_{1,N}$. Take $\varepsilon > 0$. Then by Wiener's theorem for $L_{1,N}$ [8, p. 162] there exist translates $t_1, \dots, t_k \in R_N$ and constants c_1, \dots, c_k such that

$$(4.9) \quad \int_{R_N} \left| \sum_{j=1}^K c_j \Lambda(v + t_j) - \Phi(v)g(v) \right| dv < \varepsilon.$$

Using (4.7), (4.8), and (4.9) we see that

$$(4.10) \quad \int_{R_N} \left| \sum_{j=1}^K c_j \Psi(v + t_j) \exp\{-v \cdot t_j - \frac{1}{2} t_j \cdot t_j\} - \Phi(v) \right| g(v) dv < \varepsilon.$$

Choosing h_1, \dots, h_k in the subspace of H generated by $\{\varphi_1, \dots, \varphi_N\}$ and such that $t_j = [(h_j, \varphi_1), \dots, (h_j, \varphi_N)]$ ($j=1, \dots, K$), (4.10) then implies

$$(4.11) \quad \int_B \left| \sum_{j=1}^K c_j W_{h_j} f(x) - L(x) \right| d\mu(x) < \varepsilon.$$

Hence the tame functions in $\mathcal{L}_2(\mu)$ of the form (4.5) with $m=N$ can be approximated in \mathcal{L}_1 -norm by our \mathcal{L}_1 -translates of f . Since $m=N$ represents the general case \mathfrak{B}_f , is dense in $\mathcal{L}_1(\mu)$. The proof is now complete since $\mathcal{L}_1(\mu)$ is dense in $\mathcal{L}_p(\mu)$ and the \mathcal{L}_1 -norm dominates the \mathcal{L}_p -distance, $0 < p \leq 1$.

The function $f \in \mathcal{L}_1(\mu)$ is said to be *splittable* with respect to the complete orthonormal set $\{\varphi_k\}$ if there exists a sequence of integers $N_1 < N_2 < \dots$ such that, for each integer k ,

$$(4.12) \quad f(x) = L_k(x) \cdot \Gamma_k(x) \quad \text{a.e. } [\mu]$$

where $L_k(x) = \Phi_k[(x, \varphi_1)^\sim, \dots, (x, \varphi_{N_k})^\sim]$ and Γ_k is \mathfrak{B}_k measurable on B where \mathfrak{B}_k is the minimal σ -algebra generated by the functionals $\{(\cdot, \varphi_j)^\sim, j \geq N_k + 1\}$.

REMARK 4.1. Since Γ_k is \mathfrak{B}_k -measurable on B it follows [1, p. 395] that there exists a Borel measurable function defined on the space of all real sequences such that

$$\Gamma_k(x) = F((x, \varphi_{N_k+1})^\sim, \dots) \quad \text{a.e. } [\mu].$$

Also from (4.12) and the fact that $f \in \mathcal{L}_1(\mu)$ we get that L_k, Γ_k are in $\mathcal{L}_1(\mu)$ provided $f \neq 0$.

We say that $f \in \mathcal{L}_1(\mu)$ is *negligibly split* if f is splittable and for every $\varepsilon > 0$, there exists a k such that

$$\int_B |\Gamma_k(x) - 1| d\mu(x) < \varepsilon.$$

It is easy to see that f is then the product of tame functions.

THEOREM 4.2. *Let $f \in \mathcal{L}_1(\mu)$ be negligibly split with respect to the complete orthonormal bases $\{\varphi_k\}$ in H and assume that $(\mathcal{F}_1 f)(h) \neq 0$ for all $h \in H$. Then the linear manifold \mathfrak{B} , generated by \mathcal{L}_1 -translates of f is dense in $\mathcal{L}_p(\mu)$, $0 < p \leq 1$.*

PROOF. In view of the argument given in the proof of Theorem 4.1 it suffices to prove that one can approximate in \mathcal{L}_1 -norms functions $L \in \mathcal{L}_2(\mu)$ of the form (4.5). Take $\varepsilon > 0$, and suppose $L(x)$ is given. Then there exists N_k such that $N_k \geq m$ and

$$(4.13) \quad \int_B |\Gamma_{N_k}(x) - 1| d\mu(x) < \varepsilon.$$

Again arguing as in Theorem 4.1 we can now assume that $m = N_k$. Also we have

$$(4.14) \quad \mathcal{F}_1 f(y) = \mathcal{F}_1 L_k(y) \mathcal{F}_1 \Gamma_k(y) e^{(y, y)/2}, \quad y \in H.$$

The above equation holds as indicated since the functionals L_k and Γ_k are independent (probabilistic sense). Since $\mathcal{F}_1 f(y) \neq 0$ we get $\mathcal{F}_1 L_k(y) \neq 0$ by (4.14). Now by the proof of Theorem 4.1 there exist constants c_1, c_2, \dots, c_r and vectors h_1, h_2, \dots, h_r in the subspace of H generated by $\{\varphi_1, \dots, \varphi_{N_k}\}$ such that

$$(4.15) \quad \int_B \left| \sum_{j=1}^r c_j W_{h_j} L_k(x) - L(x) \right| d\mu(x) < \varepsilon.$$

In view of Remark 4.1 it follows that

$$\Gamma_k(x + h_j) = \Gamma_k(x) \quad (j = 1, \dots, r)$$

and hence

$$W_{h_j} f(x) = [W_{h_j} L(x)] \cdot \Gamma_k(x) \quad (j = 1, \dots, r).$$

Now Γ_k and L_k are independent (probabilistic sense) thus (4.13) and (4.15) imply

$$(4.16) \quad \begin{aligned} & \int_B \left| \sum_{j=1}^r c_j W_{h_j} f(x) - \sum_{j=1}^r c_j W_{h_j} L_k(x) \right| d\mu(x) \\ &= \int_B \left| \sum_{j=1}^r c_j W_{h_j} L_k(x) [\Gamma_k(x) - 1] \right| d\mu(x) \\ &= \int_B \left| \sum_{j=1}^r c_j W_{h_j} L_k(x) \right| d\mu(x) \cdot \int_B |\Gamma_k(x) - 1| d\mu(x) \\ &\leq \left[\int_B |L(x)| d\mu(x) + \varepsilon \right] \times \varepsilon. \end{aligned}$$

Combining (4.15) and (4.16) along with $\varepsilon > 0$ being arbitrary completes the proof.

REMARK 4.2. If f is a tame function of the form (4.4) then f is easily seen to be negligibly split with respect to the complete orthonormal basis $\{\varphi_k\}$ where $\{\varphi_1, \dots, \varphi_N\}$ are as in (4.4). Thus Theorem 4.2 actually implies Theorem 4.1, but we proved both theorems since a direct proof of Theorem 4.2 would involve about the same amount of effort.

For an example of a function $f \in \mathcal{L}_1(\mu)$ which satisfies Theorem 4.2 but not Theorem 4.1 consider

$$f(x) = \exp\left\{\sum_{k=1}^{\infty} \lambda_k(x, \varphi_k)^2\right\}$$

where $\{\lambda_k\}$ is a sequence of positive numbers such that $\sum_{k=1}^{\infty} \lambda_k < \frac{1}{2}$ and $\{\varphi_k\}$ is an orthonormal set in H . Then $f \in \mathcal{L}_1(\mu)$,

$$\mathcal{F}_1 f(h) = \exp\left\{-\sum_{k=1}^{\infty} (h, \varphi_k)^2 / 2(1 - 2\lambda_k)\right\} \neq 0$$

and f is negligibly split with respect to $\{\varphi_k\}$.

If f and φ are measurable functions on B we define the *convolution of f and φ* by the usual formula

$$(4.17) \quad f * \varphi(x) = \int_B f(y)\varphi(y - x) d\mu(y).$$

In this setup convolution is not always commutative since the measure μ is not translation invariant. It does, however, act in a normal way with respect to translation if we use \mathcal{L}_1 -translates and we show this in the next lemma.

LEMMA 4.3. Suppose $f \in \mathcal{L}_1(\mu)$ and $\varphi \in \mathcal{L}_\infty(\mu)$. Then

- (1) $f * \varphi(x)$ exists for each x in B ,
- (2) $(W_h f) * \varphi(x) = f * \varphi(x + h)$ for each $x \in B$ and $h \in H$.

PROOF. Since φ is in $\mathcal{L}_\infty(\mu)$ and $\varphi(y - x)$ is measurable as a function of y for each $x \in B$ the conclusion of (1) is immediate. Now $f \in \mathcal{L}_1(\mu)$ iff $W_h f \in \mathcal{L}_1(\mu)$ for each $h \in H$ [7, p. 435] so it follows that $W_h f * \varphi(x)$ exists on B for each $x \in H$. Further,

$$\begin{aligned} W_h f * \varphi(x) &= \int_B W_h f(y)\varphi(y - x) d\mu(y) \\ &= \int_B f(y + h)\exp\left\{-(x, h)^2 - \frac{1}{2}(h, h)^2\right\}\varphi(y - x) d\mu(y) \\ &= \int_B f(y)\varphi(y - x - h) d\mu(y) = f * \varphi(x + h) \end{aligned}$$

where the third equality follows from the translation theorem [7, p. 435]. Hence (2) holds.

We now state a Tauberian theorem for abstract Wiener spaces. Its proof is exactly as in [10, p. 285] if one uses the definition of convolution in (4.17) and the \mathcal{L}_1 -translates. If φ is defined on B we say $\lim_{x \rightarrow \infty} \varphi(x) = c$ if for each $\varepsilon > 0$ there exists a bounded set E such that $|\varphi(x) - c| < \varepsilon$ on $B - E$.

THEOREM 4.3. *Suppose $f \in \mathcal{L}_1(\mu)$ and $\mathcal{F}_1 f(y) \neq 0$, $y \in H$. Further, assume $\varphi \in \mathcal{L}_\infty(\mu)$ and c is a constant such that*

$$\lim_{x \rightarrow \infty} f * \varphi(x) = c \mathcal{F}_1 f(0).$$

*Then, if f is negligibly split, we have $\lim_{x \rightarrow \infty} g * \varphi(x) = c \int_B g(x) d\mu(x)$ for all $g \in \mathcal{L}_1(\mu)$.*

As a final remark we mention that Pitt's Tauberian theorem, as it appears in [8, p. 163], can also be proved in this setting. Here, however, we would define slowly oscillating in terms of a norm bounded set and a norm bounded neighborhood of zero. Since open subsets of B have positive μ -measure the proof is as in [8].

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