BOUNDED LIMITS OF ANALYTIC FUNCTIONS

A. M. DAVIE

Abstract. Let \( U \) be a bounded open plane set, and let \( f \) be a bounded analytic function on \( U \), which is the pointwise limit of a bounded sequence \( \{f_n\} \) of uniformly continuous analytic functions. It is shown that one can find another such sequence \( \{f'_n\} \), converging to \( f \), and bounded by the supremum norm of \( f \). A similar result is proved for approximation by rational functions.

In this paper the following problem is considered: let \( U \) be a bounded open subset of the complex plane \( \mathbb{C} \), and let \( A \) be a set of bounded analytic functions on \( U \). Which bounded analytic functions on \( U \) are limits of bounded sequences of functions in \( A \) converging pointwise in \( U \)?

In the case where \( A \) consists of the polynomials this question was settled by Rubel and Shields [4], a special case had earlier been treated by Farrell [2]. A function \( f \) on \( U \) is the pointwise limit of some bounded sequence of polynomials if and only if it is the restriction to \( U \) of a bounded analytic function on \( U^* \), the Carathéodory hull of \( U \). (\( U^* \) is the interior of the complement of the unbounded component of the closure of \( U \).)

Rubel and Shields gave an example to show that the set of such pointwise limits need not be closed under uniform convergence. They constructed a set \( U \) and a sequence \( \{f_n\} \) of bounded analytic functions converging uniformly on \( U \) to \( f \), such that each \( f_n \) is a pointwise bounded limit of polynomials, but that the bounds on the approximating sequences of polynomials necessarily tended to infinity with \( n \), so that no bounded diagonal subsequence converging to \( f \) could be found. The object of this paper is to show that if \( A \) is either the algebra of uniformly continuous analytic functions on \( U \) or (under mild hypotheses on \( \partial U \)) the rational functions with poles outside \( \bar{U} \), then this phenomenon cannot occur.

The author is very grateful to T. W. Gamelin and B. K. Øksendal for many valuable conversations.

Notation. \( A(U) \) denotes the algebra of uniformly continuous analytic functions on the bounded open set \( U \) (which we regard as continuous

Received by the editors March 26, 1971.

AMS 1970 subject classifications. Primary 30A82; Secondary 30A98, 46J15.

Key words and phrases. Bounded open plane set, bounded analytic function, pointwise bounded limit, annihilating measure.

1 This work was partially supported by NSF Grant #GP-19067.
functions on the closure $\bar{U}$). For a compact plane set $X$, we denote by $R(X)$ the uniform closure on $X$ of the rational functions with poles outside $X$. $X^\circ$ and $\partial X$ denote the interior and boundary of $X$ respectively. “Measure” means “complex Borel measure” and $A(U)^\perp$ is the set of measures $\mu$ on $\bar{U}$ such that $\int f d\mu = 0$ for all $f \in A(U)$. The symbol $\|f\|$ means the supremum of $|f|$ over the domain of definition of $f$, and $\|f\|_S$ means the supremum of $|f|$ over the set $S$. Finally $H^\infty(U)$ denotes the algebra of all bounded analytic functions on $U$.

Fix a bounded open set $U$, and let $B = \{ f \in H^\infty(U) : \text{there exists a sequence } \{ f_n \} \text{ in } A(U) \text{ with } \sup_n \| f_n \| < \infty \} \text{ and } f_n \to f \text{ pointwise in } U$.\]

**Theorem 1.** Let $f \in B$. Then we can find a sequence $\{ f_n \}$, with $f_n \in A(U)$, $\| f_n \| \leq \| f \|$, converging to $f$ pointwise on $U$.

We divide the proof into two steps.

**Step 1.** Assume the theorem is false and let $\epsilon > 0$. Then we can find a sequence $\{ g_n \}$ in $A(U)$, a set $F \subseteq \bar{U}$, and a measure $\sigma \in A(U)^\perp$, such that $\| g_n \| \leq 1$, $g_n \to g$ pointwise on $U$, and $\| g \|_U < \epsilon$, $|1 - g_n| < \epsilon$ on $F$, and $\sigma(F) = 0$.

**Proof.** Assuming the theorem false we can find $f \in B$ with $\| f \| = 1$ such that, if we define

$$\lambda = \inf \{ \sup_n \| f_n \| : \{ f_n \} \text{ is a sequence in } A(U) \text{ with } f_n \to f \text{ pointwise in } U \},$$

then $\lambda > 1$.

Let $\{ f_n \}$ be a sequence in $A(U)$ with $\| f_n \| \leq \lambda$ and $f_n \to f$ pointwise. Let $m$ be a positive integer such that $\lambda^{-m} < \epsilon$ and let $\eta = \min(\lambda - 1, \lambda \epsilon/2m)$. By the definition of $\lambda$ there is a compact set $K \subseteq U$ such that $f$ is not in the closure of $T = \{ h \in A(U) : \| h \| \leq \lambda - \eta \}$ in the topology of uniform convergence on $K$. Thus we can find a measure $\mu$ on $K$ such that $\int h \, d\mu \leq 1$ for $h \in T$ but $\int f \, d\mu > 1$. The functional $h \to \int h \, d\mu$ on $A(U)$ has norm $\leq (\lambda - \eta)^{-1}$, and has a norm-preserving extension to $C(\bar{U})$ represented by a measure $\nu$, $\| \nu \| \leq (\lambda - \eta)^{-1}$.

Then $\sigma = \mu - \nu \in A(U)^\perp$. Let $G$ be a cluster point of $\{ f_n \}$ in $L^\infty(|\sigma|+|\mu|)$. Then $\int G \, d\sigma = 0$ and so $1 < \int f \, d\mu = \lim_n \int f_n \, d\mu = \lim_n \int G \, d\nu \leq \| G \| \| \nu \| \leq (\lambda - \eta)^{-1} \| G \|$. Hence $\| G \| > \lambda - \eta > 1$. Clearly $\| G \| \leq \lambda$, and $|G| \leq 1$ a.e. ($|\mu|$), so that $\lambda - \eta < |G| \leq \lambda$ on some set $F_1$ with $|\sigma| (F_1) > 0$.

The annulus $\{ \zeta : \lambda - \eta \leq |\zeta| \leq \lambda \}$ can be covered by finitely many discs with centers on $\{ |\zeta| = \lambda \}$ and radii $2\eta$. The inverse of one such disc under $G$ has positive $|\sigma|$-measure; multiplying $G$ (and $f_n$ and $f$) by a constant of modulus 1 we may assume that the center is $\lambda$.

Thus we can find a compact set $F_2 \subseteq F_1$ with $|\sigma| (F_2) > 0$ and $|\lambda - G| < 2\eta$ on $F_2$.\]
By passing to a subsequence we may assume \( f_n \to G \) weak* in \( L^\infty(|\sigma|) \), hence weakly in \( L^2(|\sigma|) \). Then we can find a sequence \( \{f'_k\} \), where each \( f'_k \) is a convex combination of functions \( f_n \), with \( f'_k \to G \) in norm in \( L^2(|\sigma|) \), and still \( f'_k \to f \) in \( U \). Again passing to a subsequence we may assume \( f'_k \to G \) a.e. (|\sigma|). By Egoroff’s theorem we can find a compact set \( F \subseteq F_\eta \) with \( \sigma(F) \neq 0 \) such that \( f'_k \to G \) uniformly on \( F \). Then we can find \( k_\eta \) so that for \( k > k_\eta \), \( |\lambda - f'_k| < 2\eta \) on \( F \).

Put \( g_k = (\lambda - f'_k)^m \) and \( g = (\lambda - f)^m \). Then \( g_k \in A(U) \), \( g_k \to g \) pointwise in \( U \), and \( \|g\| < \epsilon \).

Finally, on \( F \) we have \( |1 - g_k| = |1 - (\lambda - f'_k)^m| < \epsilon \) by the choice of \( \eta \), for \( k > k_\eta \). This completes Step 1.

**Step 2.** To prove the theorem we show that Step 1 leads to a contradiction if \( \epsilon \) is small enough. The proof is based on a construction due to Øksendal [5, Lemma 2.1], and a variation of Vituskin’s \( T_\varphi \) technique due to Gamelin.

We use \( A_1, A_2, \ldots \) to denote absolute constants. Fix \( \delta > 0 \), and choose discs \( \Delta_1, \ldots, \Delta_r \) with centers \( z_1, \ldots, z_r \) and radius \( \delta \), covering \( F \), and continuously differentiable functions \( \varphi_1, \ldots, \varphi_r \) such that:

(i) at most 25 of the discs \( \Delta_i \) meet any given point, each \( \Delta_i \) meets \( F \);
(ii) \( \varphi_i \) vanishes outside a compact subset of \( \Delta_i \), \( 0 \leq \varphi_i \leq 1 \), \( \sum_{i=1}^r \varphi_i = 1 \) on a neighborhood of \( F \), and \( |\text{grad } \varphi_i| \leq A_3 \delta^{-1} \). (For details of this construction see [3, VIII.7.1].) For \( i = 1, 2, \ldots \), \( n = 1, 2, \ldots \), define

\[
\int h_i^{(n)}(z) = \frac{1}{\pi} \int_U \frac{\varphi_i(z)}{z - \zeta} dm(z), \quad \zeta \in C,
\]

where \( m \) denotes Lebesgue measure, and \( h_i \) similarly with \( g_n \) replaced by \( g \). Then \( h_i^{(n)} \to h_i \) uniformly, since

\[
|h_i^{(n)}(\zeta) - h_i(\zeta)| \leq A_4 \min \left( \int_U \left| \frac{1}{z - \zeta} \right| dm(z) \right)
\]

the first integral is bounded by a fixed constant and the second tends to zero.

We have \( \|h_i\| \leq A_2 \|g\| < A_3 \epsilon \), so if \( n = n(\delta) \) is chosen large enough we have \( \|h_i^{(n)}\| < A_4 \epsilon \). Since \( h_i \) is analytic outside \( \Delta_i \) and vanishes at \( \infty \), in fact we have

\[
|h_i^{(n)}(\zeta)| < A_5 \epsilon \min \left( \frac{1}{|\zeta - z_i|}, \frac{\delta}{|\zeta - z_i|} \right).
\]
We observe that \( g_n q_i + h_i^{(n)} \in A(U) \) since
\[
\frac{1}{\pi} \int_{\Delta(U)} \frac{\partial q_i}{\partial \bar{z}} g_n(z) \, dm(z) \quad \text{and} \quad \frac{1}{\pi} \int_{\Delta(U)} \frac{\partial q_i}{\partial \bar{z}} g_n(z) \, dm(z)
\]
are both in \( A(U) \) (see [3, VIII, 7.1]). Define
\[
H_\delta = \sum_{i=1}^{r} (g_n q_i + h_i^{(n)})^3 \in A(U).
\]
We assert
\[
(1) \quad |H_\delta(\xi)| \leq A_3 \min(1, \frac{\delta}{d(\xi, F)}), \quad \xi \in \bar{U}.
\]
\[
(2) \quad \text{If } \xi \in F, \quad |1 - H_\delta(\xi)| < A_4 < 1 \text{ provided } \varepsilon < A_5. \quad \text{To prove this write}
\]
\[
H_\delta(\xi) = g_n \sum_{i=1}^{r} q_i^3 + 3 g_n \sum_{i=1}^{r} q_i h_i^{(n)}(g_n q_i + h_i^{(n)}) + \sum_{i=1}^{r} (h_i^{(n)})^3
\]
\[
= T_1 + T_2 + T_3 \quad \text{say.}
\]
We first estimate \( T_3 \):
\[
|T_3| \leq A_5 \varepsilon^3 \sum_{i=1}^{r} \min \left(1, \frac{\delta^3}{|\xi - z_i|^3} \right)
\]
\[
\leq A_6 \varepsilon^3 \min \left(1, \frac{\delta}{d(\xi, F)} \right)
\]
by an easy calculation, along the lines of [3, p. 212]. For any fixed \( \xi \), at most 25 of the terms summed in \( T_1 \) and \( T_2 \) are nonzero, so (1) follows.

For (2) observe that \( \|T_3\| \leq 75 A_2 \varepsilon (1 + A_2) \varepsilon \), so \( \|T_2 + T_3\| \leq A_7 \varepsilon \). Let \( \psi = \sum i q_i^3 \), since \( \sum q_i = 1 \) on \( F \), if \( \xi \in F \) then \( \psi(\xi) \geq 25^{-1} \) for some \( i \), so \( \psi(\xi) \geq 25^{-3} \). Thus, for \( \xi \in F \),
\[
|1 - H_\delta(\xi)| \leq A_7 \varepsilon + |1 - g_n^3(\xi)\psi(\xi)|
\]
\[
\leq A_7 \varepsilon + (1 - \psi(\xi)) + \psi(\xi) |1 - g_n^3(\xi)|
\]
\[
\leq 1 - 25^{-3} + (3 + A_7) \varepsilon
\]
\[
\leq 1 - \frac{25^{-3}}{2}
\]
provided
\[
\varepsilon < \frac{1}{2.25^3(3 + A_7)},
\]
which is (2).

Thus as \( \delta \to 0 \), by (1) \( H_\delta \to 0 \) boundedly on \( \bar{U} \setminus F \). Hence for each integer \( k \)
we can choose δₖ > 0 so that

\[ \left| \int_{\partial F} (1 - (1 - H_{δ_k})^k) \, dσ \right| < \frac{1}{k} . \]

Since σ ∈ A(U)⁻¹, \( \int_{\partial F} (1 - (1 - H_{δ_k})^k) \, dσ = 0 \) and so

\[ \left| \int_{F} (1 - (1 - H_{δ_k})^k) \, dσ \right| < \frac{1}{k} . \]

As \( k \to \infty \), the integrand tends uniformly to 1 on \( F \), hence \( σ(F) = 0 \), a contradiction. Theorem 1 is proved.

**Corollary.** \( B \) is closed under pointwise bounded convergence.

Next we prove an analogous result for \( R(X) \). Let \( τ \) denote plane Lebesgue measure restricted to the points of \( K \) which are not peak points for \( R(X) \). Let \( B_R \) denote the set of \( f \in L^\infty(τ) \) for which there exists a sequence \( f_n \in R(X) \) with \( \sup_n \| f_n \| < \infty \) and \( f_n \rightharpoonup f \) weak* in \( L^\infty(τ) \).

**Theorem 2.** Let \( f \in B_R \). We can find a sequence \( \{ f_n \} \) in \( R(X) \) with \( \| f_n \| \leq \| f \|_\infty \) and \( f_n \rightharpoonup f \) weak* in \( L^\infty(τ) \).

**Proof.** This is essentially the same as that of Theorem 1 so we merely indicate the necessary modifications. \( R(X) \) replaces \( A(U) \) and \( X \) replaces \( U \). \( μ \) is now a measure absolutely continuous with respect to \( τ \). Everything goes through until the construction of \( h^{(n)}_i \), now we define

\[ h^{(n)}_i(\zeta) = \frac{1}{n \zeta} \int_0^\infty \frac{\partial g_n}{\partial \zeta} \frac{g_n(z)}{z - \zeta} \, d\tau(z) . \]

The only problem is to show \( g_n \psi_i + h^{(n)}_i \in R(X) \), i.e. that

\[ F(\zeta) = \int_0^\infty \frac{\partial g_n}{\partial \zeta} \frac{g_n(z)}{z - \zeta} \, dm(z) \]

is in \( R(X) \), where \( \partial \) is the set of peak points. But if \( \theta \perp R(X) \) then

\[ \int_X F(\zeta) \, d\theta(\zeta) = \int_0^\infty \int_X \frac{\partial g_n}{\partial \zeta} \frac{g_n(z)}{z - \zeta} \, d\theta(\zeta) \, dm(z) \]

\[ = \int_0^\infty \frac{\partial g_n}{\partial \zeta} \left( \int_X \frac{1}{z - \zeta} \, d\theta(\zeta) \right) \, dm(z) \]

the integrals converging absolutely since \( 1/|z| \) is integrable over any bounded set. If \( zeX \) is such that \( \int (1/|z - \zeta|)|d\theta(\zeta)| < \infty \) and \( \int (1/(z - \zeta)) \, d\theta(\zeta) \neq 0 \) then \( z \) is not a peak point for \( R(X) \) by the proof of
Theorem II.8.5 of [3]. Hence \( \int_X \frac{1}{(z-\zeta)} \, d\theta(\zeta) = 0 \) for almost all \( z \in \partial \), so that \( \int F \, d\theta = 0 \). Hence \( F \in R(X) \).

The rest of the proof works as before.

Notes. (1) If \( \{f_n\} \) is a bounded sequence in \( R(X) \) converging weak* in \( L^\infty(\tau) \) to \( f \in B_R \), then in fact \( f_n \) converges pointwise on \( X \setminus \partial \) to \( g \) say, where \( g = f \) a.e. (\( \tau \)) and \( g \) depends only on \( f \) (not on the choice of \( \{f_n\} \)). To see this choose a sequence \( \{g_n\} \) of convex combinations of the functions \( f_n \) converging a.e. (\( \tau \)) to \( f \). Let \( z \in X \setminus \partial \), let \( \varepsilon > 0 \) be given, and choose \( \zeta \in K \setminus \partial \) so that \( g_n(z) \rightarrow f(\zeta) \) and \( \|z - \zeta\| < \delta \) where \( z, \zeta \) are the evaluation functionals on \( R(X) \). (This is possible by [1, Theorem 2].) Then

\[
|g_n(z) - g_n(z)| < M \varepsilon \quad \text{where} \quad M = \sup \|f_n\|.
\]

Since \( \varepsilon \) is arbitrary we deduce that \( g_n(z) \) converges to a limit \( g(z) \) such that \( |g(z) - f(\zeta)| < M \varepsilon \) for almost all \( \zeta \) satisfying \( \|z - \zeta\| < \delta \). Then \( g_n \) converges pointwise to \( g \) on \( X \setminus \partial \) and \( g \) is determined by \( f \). It follows that the original sequence \( \{f_n\} \) converges pointwise to \( g \) on \( X \setminus \partial \).

(2) The question naturally arises as to whether one can obtain a similar result involving convergence only on \( X^0 \). If almost all points of \( \partial X \) are peak points then Theorem 2 answers this question. On the other hand one can easily construct examples to show that some restriction is necessary. In general we have the following: let \( f \in H^\infty(X^0) \) be the limit of a bounded sequence \( \{f_n\} \) in \( R(X) \). Then we can find a subsequence converging pointwise on \( X \setminus \partial \) to \( g \) say, so that \( g|X^0 = f \). If \( R(\partial K) = C(\partial K) \) then \( g \) depends only on \( f \) (for if \( f = 0 \) then \( f_n \rightarrow 0 \) pointwise in \( X \setminus \partial \)).

\[
f_n(z) - f_n(z_0) \xrightarrow{z \to z_0} g(z) \quad z, z_0 \in B_R
\]

for each \( z_0 \in X^0 \), whence \( h g \in B_R \) for all \( h \in C(X) \), which implies \( g = 0 \) on \( X \setminus \partial \) in view of the inequality

\[
|g(z) - g(\zeta)| \leq M \|z - \zeta\|, \quad z, \zeta \in X \setminus \partial, \quad M = \sup \|f_n\|.
\]

We conjecture that in this case \( \|g\| = \|f\| \), which is equivalent to the analogue of Theorem 2 for convergence on \( X^0 \). A more concrete way of stating this conjecture is as follows: suppose \( R(\partial X) = C(\partial X) \). There exists \( \varepsilon > 0 \) such that if \( \{f_n\} \) is a sequence in \( R(X) \) with \( \|f_n\| \leq 1, f_n \to f \) on \( X^0 \) with \( \|f\| < \varepsilon \), and \( z \) is a point in \( \partial X \) with \( |1 - f_n(z)| < \varepsilon \) for each \( n \), then \( z \) is a peak point for \( R(X) \).

The author is indebted to T. W. Gamelin for the idea of considering convergence on the nonpeak points.

References


Department of Mathematics, University of California, Los Angeles, California 90024

Current address: Mathematical Institute, 20 Chambers Street, Edinburgh, Scotland