ON INTEGRABLE AND BOUNDED AUTOMORPHIC FORMS. II

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Abstract. For a finitely generated Fuchsian group, every integrable automorphic form of arbitrary real dimension $<-2$ is bounded. If the group is, in addition, of second kind, then every integrable automorphic form of dimension $-2$, with arbitrary factors of automorphy, is zero.

1. Introduction. Throughout, $\Gamma$ denotes a Fuchsian group acting on the unit disc $U$ of the complex plane. For any given real number $q$, we choose and fix, once and for all, a system $\rho(q, T, z)$ ($z \in U, T \in \Gamma$) of factors of automorphy of dimension $-2q$ belonging to $\Gamma$ (cf. [3]). Note that, if $q$ is an integer, $\rho(q, T, z) = \chi(T)T'(z)^q$, where $\chi$ is a character of $\Gamma$.

Let $\Omega$ be a Poincaré normal polygon of $\Gamma$ centered at a point in $U$ not fixed by $\Gamma$. Denote by $A_q(\Gamma)$ the space of integrable, holomorphic automorphic forms of dimension $-2q$, and, by $B_q(\Gamma)$, the space of bounded, holomorphic automorphic forms of dimension $-2q$ (cf. [3]). We prove:

Theorem 1. Let $\Gamma$ be finitely generated. Then $A_q(\Gamma) \subset B_q(\Gamma)$ for $q > 1$.

Theorem 2. Let $\Gamma$ be finitely generated and be of second kind. Then $A_1(\Gamma) = \{0\}$.

Remarks. In the case where $q$ is an integer and $\rho(q, T, z)$ are the standard factors of automorphy $T'(z)^q$, Theorem 1 was proved by Drasin and Earle [1] by an entirely different method. For arbitrary real $q$ and arbitrary $\rho$, Theorem 1 was established in [3] under the additional hypothesis that $\Gamma$ contains no parabolic elements. Theorem 2 is proved here by reducing it to the already established (cf. [3]) special case of standard factors of automorphy $\rho(1, T, z) = T'(z)$. Thus, the present note is in the nature of an addendum to [3].
2. Proof of Theorem 1. For $q>1$ and $z, \zeta$ in $U$, let

$$K(z, \zeta) = \pi^{-1}(2q - 1)(1 - |z\zeta|)^{-2q},$$

where $K$ is analytic in $z$ and $K(0, \zeta)>0$. By Theorem 1 of [3], it suffices to show that

$$(2.1) \sup_{z \in U} (1 - |z|^{2q})\alpha(z, z) < \infty,$$

where

$$\alpha(z, \zeta) \equiv \sum_{T \in \Gamma} \rho(q, T, z)K(Tz, \zeta).$$

Let $H^\infty$ be the Banach space of bounded analytic functions on $U$ with the supremum norm and $E$ be the Banach space of all holomorphic automorphic forms $F$ of dimension $-2q$ such that the norm

$$\|F\| = \sup_{z \in U} |F(z)| < \infty.$$

For $f$ in $H^\infty$, consider the Poincaré series

$$\theta f(z) = \sum_{T \in \Gamma} \rho(q, T, z)f(Tz).$$

Knopp [2, Proposition 3] proved that, if $f \in H^\infty$, then $\theta f \in E$. Standard arguments show that convergence of a sequence in norm in either of the spaces $H^\infty$ and $E$ implies uniform convergence on compact subsets of $U$. This readily implies that the map $f \mapsto \theta f$ is a closed linear map of $H^\infty$ into $E$. The closed graph theorem then implies that $\theta : H^\infty \rightarrow E$ is a bounded linear map, i.e., there exists a constant $M < \infty$ such that, for all $f$ in $H^\infty$,

$$|\theta f(z)| \leq M \sup_{t \in U} |f(t)|, \quad z \in \Omega.$$

Applying this to $f(t) = K(t, \zeta)$, we conclude that

$$|\alpha(z, \zeta)| \leq M \pi^{-1}(2q - 1) \frac{2^{2q}}{(1 - |z\zeta|)^2q}, \quad z \in \Omega, \zeta \in \Omega.$$

Setting $\zeta = z$, we obtain the existence of a constant $N < \infty$ such that

$$\beta(z) \equiv \alpha(z, z)(1 - |z|^2)^{2q} \leq N, \quad z \in \Omega.$$

Since, for all $T$ in $\Gamma$, $\beta \circ T = \beta$ and $\Omega$ is a fundamental region for $\Gamma$, (2.1) follows and the theorem is proved.

3. An auxiliary result. The proof of Theorem 2 is based on the following result of independent interest.
Theorem 3. Let $\Gamma$ be finitely generated and be of second kind. For any character $\lambda$ of $\Gamma$, there exists a function $g$ bounded and analytic on $U$, not identically zero on $U$ and satisfying

\begin{equation}
\lambda(T) \cdot (g \circ T) = g, \quad T \in \Gamma.
\end{equation}

Proof. If the theorem is true for $\Gamma$, then it is true for every conjugate of $\Gamma$ (in the full group of conformal selfmaps of $U$). Hence we can and do assume that the origin in $U$ is not fixed by $\Gamma$ and let $\Omega$ be the normal polygon centered at the origin.

Let $f$ be a holomorphic automorphic form of dimension $-2$ belonging to $\Gamma$ with factors of automorphy $\lambda(T) \cdot T'(z)$:

\begin{equation}
\lambda(T) \cdot T' \cdot (f \circ T) = f, \quad T \in \Gamma.
\end{equation}

Let $g$ be the antiderivative of $f$ satisfying $g(0)=0$. (3.2) implies, for each $T$ in $\Gamma$, the existence of a constant $C(T,f)$ such that

\begin{equation}
\lambda(T) \cdot (g \circ T) = g + C(T,f), \quad T \in \Gamma.
\end{equation}

It is readily verified that, for all $T_1, T_2$ in $\Gamma$,

\begin{equation}
C(T_1T_2,f) = \lambda(T_2) \cdot C(T_1,f) + C(T_2,f).
\end{equation}

It follows that the set $\{T \in \Gamma | C(T,f)=0\}$ is a subgroup of $\Gamma$. Since $\Gamma$ is finitely generated, the theorem will be proved if there exists a nontrivial $f$ such that the corresponding $g$ is bounded on $U$ and is such that

\begin{equation}
C(S_j,f) = 0, \quad j = 1, 2, \ldots, m,
\end{equation}

where $\{S_1, \ldots, S_m\}$ is a generating set for $\Gamma$. We choose $f(z)=\theta p(z)=\sum_{T \in \Gamma} \lambda(T) T'(z) p(Tz)$ for a suitable polynomial $p=\sum_{i=0}^{n} a_i p_i$, where $p_i(z)=z^i$ and $n \geq m$. Let $f_1=\theta p$. Since $\theta$ is a linear map and $h \rightarrow C(T,h)$ is linear, the condition (3.3) is equivalent to: $\sum_{i=0}^{n} a_i C(S_j,f_i)=0$, $j=1, 2, \ldots, m$. These are $m$ linear equations in $n+1>m$ unknowns $a_i$ and hence they have a nontrivial solution $(a_0, \ldots, a_n)$. The corresponding $f$ satisfies (3.3). Also, remembering that $\Gamma$ is of the second kind and examining the behaviour of $f$ at the point at infinity (cf. [4, Proposition 3]), one sees that $f$ is not identically zero on $U$. Hence the corresponding $g \neq 0$ on $U$ and satisfies (3.1). Since $p$ is bounded on $U$, Proposition 3 of Knopp [2] shows that $f=\theta p$ is bounded on $\Omega: \sup_{z \in \Omega} |f(z)| \leq M < \infty$. Hence, for $z$ in $\Omega$,

$$
|g(z)| = \left| \int_0^\infty f(t) \, dt \right| \leq M |z| \leq M;
$$

here, we have integrated along the radial segment and this is justified.
since \( \Omega \) is non-Euclidean convex and the radial segment is the non-Euclidean line segment. Thus \( g \) is bounded on \( \Omega \). Since \( \Omega \) is a fundamental region for \( \Gamma \) and \( |\lambda|=1 \), (3.1) now implies that \( g \) is bounded on \( U \) thus completing the proof.

4. **Proof of Theorem 2.** Let \( F \in A_1(\Gamma) \) and the factors of automorphy be \( \rho(1, z, T) = \chi(T) T'(z) \), where \( \chi \) is a character of \( \Gamma \). Thus \( \chi(T) T'(z) \cdot (F \circ T) = F \) and \( \int_{\Omega} |F(z)| \, dx \, dy/(1-|z|^2) < \infty \). Choose \( g \) as in Theorem 3 above with \( \lambda = \chi \) and let \( G = g \cdot F \). Then \( T'(G \circ T) = G \) for all \( T \) in \( \Gamma \) and thus \( G \) is a holomorphic automorphic form of dimension \(-2\) with the standard factors of automorphy \( T'(z) \). Moreover \( G \) is an integrable form:

\[
\int_{\Omega} \int |G(z)| \cdot \frac{dx \, dy}{1-|z|^2} \leq \sup_{z \in U} |g(z)| \int_{\Omega} \int \frac{|F(z)|}{1-|z|^2} \, dx \, dy < \infty.
\]

Hence, by Theorem 3 of [3], \( G \) is identically zero on \( U \). Since \( g \) is not identically zero, it follows that \( F \) is identically zero on \( U \), thus establishing the theorem.

**References**


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