PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 32, Number 1, March 1972

ON INTEGRABLE AND BOUNDED AUTOMORPHIC FORMS. II

T. A. METZGER AND K. V. RAJESWARA RAO[1]

ABSTRACT. For a finitely generated Fuchsian group, every integrable automorphic form of arbitrary real dimension <-2 is bounded. If the group is, in addition, of second kind, then every integrable automorphic form of dimension -2, with arbitrary factors of automorphy, is zero.

1. **Introduction.** Throughout, Γ denotes a Fuchsian group acting on the unit disc U of the complex plane. For any given real number q, we choose and fix, once and for all, a system $\rho(q, T, z)$ $(z\in U,\ T\in\Gamma)$ of factors of automorphy of dimension $-2q$ belonging to Γ (cf. [3]). Note that, if q is an integer, $\rho(q, T, z)=\chi(T)T'(z)^q$, where χ is a character of Γ.

Let Ω be a Poincaré normal polygon of Γ centered at a point in U not fixed by Γ. Denote by $A_q(\Gamma)$ the space of *integrable*, holomorphic automorphic forms of dimension $-2q$, and, by $B_q(\Gamma)$, the space of *bounded*, holomorphic automorphic forms of dimension $-2q$ (cf. [3]). We prove:

THEOREM 1. *Let Γ be finitely generated. Then $A_q(\Gamma)\subset B_q(\Gamma)$ for $q>1$.*

THEOREM 2. *Let Γ be finitely generated and be of second kind. Then* $A_1(\Gamma)=\{0\}$.

REMARKS. In the case where q is an integer and $\rho(q, T, z)$ are the standard factors of automorphy $T'(z)^q$, Theorem 1 was proved by Drasin and Earle [1] by an entirely different method. For arbitrary real q and arbitrary ρ, Theorem 1 was established in [3] under the additional hypothesis that Γ contains no parabolic elements. Theorem 2 is proved here by reducing it to the already established (cf. [3]) special case of standard factors of automorphy $\rho(1, T, z)=T'(z)$. Thus, the present note is in the nature of an addendum to [3].

Received by the editors May 20, 1971.

AMS 1969 *subject classifications*. Primary 3049; Secondary 3045, 3085.

Key words and phrases. Fuchsian group, finitely generated, factors of automorphy, character, automorphic forms, bounded, integrable.

[1] The authors were supported, in part, by the National Science Foundation.

© American Mathematical Society 1972

2. **Proof of Theorem 1.** For $q>1$ and z, ζ in U, let

$$K(z, \zeta) = \pi^{-1}(2q-1)(1-z\bar{\zeta})^{-2q},$$

where K is analytic in z and $K(0, \zeta)>0$. By Theorem 1 of [3], it suffices to show that

$$\sup_{z\in U}(1-|z|^2)\alpha(z, z) < \infty, \tag{2.1}$$

where

$$\alpha(z, \zeta) \equiv \sum_{T\in\Gamma} \rho(q, T, z)K(Tz, \zeta).$$

Let H^∞ be the Banach space of bounded analytic functions on U with the supremum norm and E be the Banach space of all holomorphic automorphic forms F of dimension $-2q$ such that the norm

$$\|F\| = \sup_{z\in\Omega} |F(z)| < \infty.$$

For f in H^∞, consider the Poincaré series

$$\theta f(z) = \sum_{T\in\Gamma} \rho(q, T, z)f(Tz).$$

Knopp [2, Proposition 3] proved that, if $f\in H^\infty$, then $\theta f\in E$. Standard arguments show that convergence of a sequence in norm in either of the spaces H^∞ and E implies uniform convergence on compact subsets of U. This readily implies that the map $f\to\theta f$ is a closed linear map of H^∞ into E. The closed graph theorem then implies that $\theta: H^\infty\to E$ is a bounded linear map, i.e., there exists a constant $M<\infty$ such that, for all f in H^∞,

$$|\theta f(z)| \leqq M \sup_{t\in U} |f(t)|, \qquad z\in\Omega.$$

Applying this to $f(t)=K(t, \zeta)$, we conclude that

$$|\alpha(z, \zeta)| \leqq M\pi^{-1}(2q-1)\frac{2^{2q}}{(1-|\zeta|^2)^{2q}}, \qquad z\in\Omega, \zeta\in\Omega.$$

Setting $\zeta=z$, we obtain the existence of a constant $N<\infty$ such that

$$\beta(z) \equiv \alpha(z, z)(1-|z|^2)^{2q} \leqq N, \qquad z\in\Omega.$$

Since, for all T in Γ, $\beta\circ T=\beta$ and Ω is a fundamental region for Γ, (2.1) follows and the theorem is proved.

3. **An auxiliary result.** The proof of Theorem 2 is based on the following result of independent interest.

THEOREM 3. *Let Γ be finitely generated and be of second kind. For any character λ of Γ, there exists a function g bounded and analytic on U, not identically zero on U and satisfying*

$$\lambda(T)\cdot(g\circ T)=g,\qquad T\in\Gamma. \tag{3.1}$$

PROOF. If the theorem is true for Γ, then it is true for every conjugate of Γ (in the full group of conformal selfmaps of U). Hence we can and do assume that the origin in U is not fixed by Γ and let Ω be the normal polygon centered at the origin.

Let f be a holomorphic automorphic form of dimension -2 belonging to Γ with factors of automorphy $\lambda(T)\cdot T'(z)$:

$$\lambda(T)\cdot T'\cdot(f\circ T)=f,\qquad T\in\Gamma. \tag{3.2}$$

Let g be the antiderivative of f satisfying $g(0)=0$. (3.2) implies, for each T in Γ, the existence of a constant $C(T,f)$ such that

$$\lambda(T)\cdot(g\circ T)=g+C(T,f),\qquad T\in\Gamma.$$

It is readily verified that, for all T_1, T_2 in Γ,

$$C(T_1T_2,f)=\lambda(T_2)\cdot C(T_1,f)+C(T_2,f).$$

It follows that the set $\{T\in\Gamma | C(T,f)=0\}$ is a subgroup of Γ. Since Γ is finitely generated, the theorem will be proved if there exists a nontrivial f such that the corresponding g is bounded on U and is such that

$$C(S_j,f)=0,\qquad j=1,2,\cdots,m, \tag{3.3}$$

where $\{S_1,\cdots,S_m\}$ is a generating set for Γ. We choose $f(z)=\theta p(z)\equiv \sum_{T\in\Gamma}\lambda(T)T'(z)p(Tz)$ for a suitable polynomial $p=\sum_{i=0}^{n}a_ip_i$, where $p_i(z)=z^i$ and $n\geqq m$. Let $f_i=\theta p_i$. Since θ is a linear map and $h\to C(T,h)$ is linear, the condition (3.3) is equivalent to: $\sum_{i=0}^{n}a_iC(S_j,f_i)=0$, $j=1,2,\cdots,m$. These are m linear equations in $n+1>m$ unknowns a_i and hence they have a nontrivial solution $(a_0,\cdots,a_n)$. The corresponding f satisfies (3.3). Also, remembering that Γ is of the second kind and examining the behaviour of f at the point at infinity (cf. [**4**, Proposition 3]), one sees that f is not identically zero on U. Hence the corresponding $g\not\equiv 0$ on U and satisfies (3.1). Since p is bounded on U, Proposition 3 of Knopp [**2**] shows that $f=\theta p$ is bounded on Ω: $\sup_{z\in\Omega}|f(z)|\equiv M<\infty$. Hence, for z in Ω,

$$|g(z)|=\left|\int_0^z f(t)\,dt\right|\leqq M\,|z|\leqq M;$$

here, we have integrated along the radial segment and this is justified since Ω is non-Euclidean convex and the radial segment is the non-Euclidean line segment. Thus g is bounded on Ω. Since Ω is a fundamental region for Γ and $|\lambda|=1$, (3.1) now implies that g is bounded on U thus completing the proof.

4. **Proof of Theorem 2.** Let $F\in A_1(\Gamma)$ and the factors of automorphy be $\rho(1, z, T)=\chi(T)T'(z)$, where χ is a character of Γ. Thus $\chi(T)\cdot T'\cdot(F\circ T)=F$ and $\iint_\Omega |F(z)|\,dx\,dy/(1-|z|^2)<\infty$. Choose g as in Theorem 3 above with $\lambda=\bar{\chi}$ and let $G=g\cdot F$. Then $T'\cdot(G\circ T)=G$ for all T in Γ and thus G is a holomorphic automorphic form of dimension -2 with the *standard* factors of automorphy $T'(z)$. Moreover G is an integrable form:

$$\iint_\Omega |G(z)|\cdot\frac{dx\,dy}{1-|z|^2} \leqq \sup_{z\in U}|g(z)| \iint_\Omega \frac{|F(z)|}{1-|z|^2}\,dx\,dy < \infty.$$

Hence, by Theorem 3 of [3], G is identically zero on U. Since g is not identically zero, it follows that F is identically zero on U, thus establishing the theorem.

References

1. D. Drasin and C. J. Earle, *On the boundedness of automorphic forms*, Proc. Amer. Math. Soc. **19** (1968), 1039–1042. MR **39** #442.

2. M. I. Knopp, *A corona theorem for automorphic forms and related results*, Amer. J. Math. **91** (1969), 599–618. MR **40** #4450.

3. T. A. Metzger and K. V. Rajeswara Rao, *On integrable and bounded automorphic forms*, Proc. Amer. Math. Soc. **28** (1971), 562–566.

4. K. V. Rajeswara Rao, *Fuchsian groups of convergence type and Poincaré series of dimension* -2, J. Math. Mech. **18** (1968/69), 629–644. MR **38** #6065.

Department of Mathematics, Purdue University, Lafayette, Indiana 47907

Current address (Metzger): Department of Mathematics, Texas A & M University, College Station, Texas 77843