TIME-VARIABLE SINGULARITIES FOR SOLUTIONS OF THE HEAT EQUATION

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Abstract. A solution \( u(x, t) \) of the two-dimensional heat equation \( u_{xx} = u \), may have the representation

\[
u(x, t) = \int_{-\infty}^{\infty} k(x - y, t) \, d\alpha(y)\]

where \( k(x, t) = (4\pi t)^{-1/2} \exp\left[-x^2/(4t)\right] \), valid in some strip \( 0 < t < c \) of the \( x, t \)-plane. If so, \( u(x_0, t) \) is known to be an analytic function of the complex variable \( t \) in the disc \( \text{Re}(1/t) > 1/c \), for each fixed real \( x_0 \). It is shown here that if \( \alpha(y) \) is nondecreasing and not absolutely continuous then \( u(x_0, t) \) must have a singularity at \( t = 0 \).

Examples show that both restrictions on \( \alpha(y) \) are necessary for that conclusion. It is shown further under the same hypothesis on \( \alpha(y) \), that for each fixed positive \( t_0 < c \), \( u(x, t_0) \) is an entire function of \( x \) of order 2 and of type \( 1/(4t_0) \). Compare the function \( k(x, t) \) itself for a check on both conclusions.

1. Introduction. In a recent note [1969] H. Pollard and I discussed the representation of a function \( f(x) \) by the Gaussian integral

\[
f(x) = \int_{-\infty}^{\infty} k(x - y, a) \, d\alpha_a(y),
\]

where \( \alpha_a(y) \) is a distribution function and \( k(x, t) \) is the fundamental solution of the heat equation,

\[
k(x, t) = (4\pi t)^{-1/2} \exp\left[-x^2/(4t)\right].
\]

We showed that if \( f(x) \) has one such representation for a given constant \( a \), then there is always a finite maximum value \( a = b \) for which (1.1) holds. In fact, Pollard in an earlier note (unpublished) gave a precise formula for \( b \) in the special case in which the corresponding distribution function \( \alpha_b(y) \) is not absolutely continuous. It was

\[
1/b = \text{l.u.b.} \limsup_{n \to \infty} \frac{1}{n} \left| \frac{1}{2n} f^{(2n)}(x) \right|^{1/n}.
\]

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The present note evolved from my effort to simplify this formula. I show that the symbol l.u.b. may be omitted because what follows it is independent of x. I prove in fact that \( f(x) \), as defined by (1.1) with \( \alpha(y) \) a non-absolutely continuous distribution function, is entire of order 2 and of type \( 1/(4a) \). Since order and type are independent of origin we have from a familiar formula (R. P. Boas [1954, p. 11])

\[
\limsup_{n \to \infty} (1/n) \left| f^{(n)}(x) \right|^{2/n} = 2/(4ae).
\]

We show that this equation remains true for the present function \( f(x) \) when \( n \) is replaced by \( 2n \) and thus obtain the desired simplification of Pollard’s formula.

The above conclusion about the order and type of \( f(x) \) really comes as a corollary to the principal theorem of this note, one which seems to have independent significance in the theory of heat conduction. I prove that if \( u(x, t) \) is a solution of the heat equation

(1.2) \[ \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \]

which has the representation

\[
u(x, t) = \int_{-\infty}^{\infty} k(x - y, t) \, d\alpha(y),
\]

where \( \alpha(y) \) is a nondecreasing and nonabsolutely continuous function, then \( u(x, t) \), considered as a function of the complex variable \( t \), has a singularity at \( t=0 \) for every real \( x \). A case in point is \( k(x, t) \) itself, for which \( \alpha(y) \) is a step-function with a single positive jump. The exponential factor in \( k(x, t) \) has an essential singularity at \( t=0 \) for every \( x \) except \( x=0 \), and even then the other factor has the predicted singularity. For the validity of the theorem, the hypothesis that \( \alpha(y) \) is not absolutely continuous is essential. The simplest examples show this. Indeed if \( \alpha(y)=y \) then \( u(x, t) \) is identically equal to unity. In §4 appear further examples to show this point and another to show that the nondecreasing character of \( \alpha(y) \) cannot be omitted in the hypothesis either.

2. Time-variable singularities of temperature functions. We now state our principal result as a theorem.

**Theorem 1.** If

(2.1) \[ u(x, t) = \int_{-\infty}^{\infty} k(x - y, t) \, d\alpha(y), \quad 0 < t < c, \]

where \( \alpha(y) \) is nondecreasing and not absolutely continuous, then for every real \( x_0 \) the function \( u(x_0, t) \) is analytic in the disc \( \text{Re}(1/t) > 1/c \) of the complex \( t \)-plane and has a singularity at \( t=0 \).
Since

\[ u(x - x_0, t) = \int_{-\infty}^{\infty} k(x - y, t) \, dx(y - x_0), \]

it is only necessary to consider the case \( x_0 = 0 \) in (2.1). That equation is then equivalent to

\[ (4\pi t)^{1/2} u(0, t) = \int_0^\infty \exp[-y^2/(4t)] \, d[x(y) - x(-y)]. \]

Since this integral becomes a Laplace transform in the variable \( 1/(4t) \) after \( y^a \) is replaced by a new variable of integration, the stated analyticity of \( u(0, t) \) is apparent. The disc in question has its center at the real point \( t = c/2 \) of the complex \( t \)-plane and has radius \( c/2 \).

To prove that \( t = 0 \) is a singularity we assume the contrary and seek a contradiction. That is, we assume

\[ u(0, t) = \sum_{n=0}^\infty a_n t^n, \quad |t| < \rho. \]

I have proved elsewhere (D. V. Widder [1970]) that any such function has the integral representation

\[ u(0, t) = \int_0^\infty k(y, t) \varphi(y) \, dy, \]

where \( \varphi(y) \) is an even entire function. We sketch the proof, which depends on the identity

\[ (2n)! \, t^n = 2(n!) \int_0^\infty k(y, t) y^{2n} \, dy, \quad t > 0. \]

Combining (2.4) and (2.2) we obtain (2.3) if

\[ \varphi(y) = 2 \sum_{n=0}^\infty n! \, a_n \frac{y^{2n}}{(2n)!}. \]

The term-by-term integration is justified by use of the relation

\[ \lim sup |a_n|^{1/n} \leq 1/\rho, \]

assumed in (2.2). This relation also shows that \( \varphi(y) \) is entire.

The two integral representations for \( u(0, t) \) and the uniqueness theorem for Laplace transforms now insure that

\[ x(y) - x(-y) = \int_0^y \varphi(r) \, dr \]

for \( y \geq 0 \). Since both sides of this equation clearly represent odd functions it must also hold for \(-\infty < y < \infty \). From (2.5) we have for any \( h > 0 \) and
any $y_0$ that
\[ [\alpha(y_0 + h) - \alpha(y_0)] + [\alpha(-y_0) - \alpha(-y_0 - h)] = \int_{y_0}^{y_0 + h} \varphi(r) \, dr. \]
Since $\alpha(y)$ is nondecreasing it follows that
\[ 0 \leq \alpha(y_0 + h) - \alpha(y_0) \leq \int_{y_0}^{y_0 + h} \varphi(r) \, dr, \]
\[ 0 \leq \alpha(-y_0) - \alpha(-y_0 - h) \leq \int_{y_0}^{y_0 + h} \varphi(r) \, dr. \]
Allowing $h$ to approach zero we see that $\alpha(y_0 +) = \alpha(y_0)$ and $\alpha(-y_0) = \alpha(-y_0 -)$. Since $y_0$ is arbitrary it is seen that $\alpha(y)$ is continuous for $-\infty < y < \infty$. Moreover,
\[ 0 \leq \frac{\alpha(y_0 + h) - \alpha(y_0)}{h} \leq \frac{1}{h} \int_{y_0}^{y_0 + h} \varphi(r) \, dr, \quad 0 \leq D_+ \alpha(y_0) \leq \varphi(y_0). \]
That is, one of the Dini derivatives is finite at every point. This fact with the continuity of $\alpha(y)$ guarantees that $\alpha(y)$ is absolutely continuous. See, for example, S. Saks [1937, Theorem 4.6 on p. 271 and Theorem 6.7 on p. 227]. But this contradicts a hypothesis of the theorem, as desired.

3. **Consequences of the main theorem.** The previous result has immediate consequences for functions $f(x)$ which have the integral representation (1.1).

**Theorem 2.** If
\[ f(x) = \int_{-\infty}^{\infty} k(x - y, c) \, d\alpha(y), \]
where $\alpha(y)$ is nondecreasing, bounded and not absolutely continuous, then for all real $x$ and for $|t| < c$,
\[ \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} f^{(2n)}(x) = \int_{-\infty}^{\infty} k(x - y, c - t) \, d\alpha(y). \] (3.1)

The function
\[ u(x, t) = \int_{-\infty}^{\infty} k(x - y, t) \, d\alpha(y) \] (3.2)
satisfies the hypotheses of Theorem 1 for every positive $c$ since the integral (3.2) clearly converges for all positive $t$. Hence by that theorem $u(x_0, t)$ is analytic in the complex half-plane $\Re t > 0$. Consequently the Taylor expansion
\[ u(x_0, c - t) = \sum_{n=0}^{\infty} \frac{\partial^n u(x_0, c)(-t)^n}{n!} \] (3.3)
is valid for $|t|<c$. But since $u(x, t)$ satisfies the heat equation (1.2) we see that

$$\frac{\partial^n}{\partial t^n} u(x_0, c) = \frac{\partial^{2n}}{\partial x^{2n}} u(x_0, c) = f^{(2n)}(x_0),$$

and equation (3.3) is equivalent to (3.1).

**Corollary 2.1.** If $f(x)$ is defined as in Theorem 2, then

$$\limsup_{n \to \infty} \frac{|f^{(2n)}(x)|^{1/n}}{n!^{1/n}} = \frac{1}{ce}, \quad -\infty < x < \infty. \quad (3.4)$$

For, by Theorem 1, the series (3.1) defines a function of $t$ which has a singularity at $t=c$. Hence its radius of convergence is $c$ for every real $x$. That is,

$$\limsup_{n \to \infty} \left[ \frac{|f^{(2n)}(x)|}{n!} \right]^{1/n} = \frac{1}{c}.$$  

Stirling's formula now yields (3.4).

**Corollary 2.2.** If $f(x)$ is defined as in Theorem 2, then

$$\limsup_{n \to \infty} (1/n) |f^{(n)}(x)|^{2/n} = 1/(2ce). \quad (3.5)$$

so that $f(x)$ is entire of order 2 and of type $1/(4c)$.

By familiar properties of the Weierstrass transform we have from equation (3.2) that

$$\frac{\partial}{\partial x} u(x, t) = \int_{-\infty}^{\infty} \frac{\partial}{\partial x} k(x - y, t) \, dx(y),$$

valid for $\Re t>0$. As before, the Taylor expansion

$$\frac{\partial}{\partial x} u(x_0, c - t) = \sum_{n=0}^{\infty} \frac{\partial^{n+1}}{\partial x \partial t^n} u(x_0, c) \frac{(-t)^n}{n!} = \sum_{n=0}^{\infty} f^{(2n+1)}(x_0) \frac{(-t)^n}{n!}$$

must hold at last for $|t|<c$. Hence

$$\limsup_{n \to \infty} \left| \frac{f^{(2n+1)}(x_0)}{n!} \right|^{1/n} \leq \frac{1}{c}.$$  

This is equivalent to

$$\limsup_{n \to \infty} 1/(2n+1) |f^{(2n+1)}(x_0)|^{2/(2n+1)} \leq 1/(2ce).$$

This inequality combined with equation (3.4) yields equation (3.5), and the proof is complete. Unlike the function $u(x, t)$ of equation (3.2) its
derivative (3.6) need not be singular at $t=0$ for all $x$. Indeed if $\alpha(y)$ is odd, the derivative is identically zero when $x=0$.

4. Examples. In the introduction we observed that for the validity of Theorem 1 it is essential that $\alpha(y)$ should not be absolutely continuous. We give further examples here to show this.

Example A.

$$u(x, t) = \int_0^\infty k(x - y, t) \, dy = \frac{1}{2} \text{erfc}(-x/(4t)^{1/2}).$$

Here $\alpha(y)$ is nondecreasing and absolutely continuous. The function $u(x, t)$ is singular at $t=0$ for every $x$ except $x=0$ when $u(0, t) \equiv \frac{1}{2}$.

Example B.

$$u(x, t) = \int_{-\infty}^\infty k(x - y, t) \exp[-y^2/(1 + 4t^2)] \, dy = \exp[-x^2/(1 + 4t^2)]/(1 + 4t^2)^{1/2}.$$

Again $\alpha(y)$ is nondecreasing and absolutely continuous, even bounded. The function $u(x, t)$ is not singular at $t=0$ for any $x$. Note that this type of situation always obtains when $\alpha(y)$ is entire of order 2 and of finite type. For then $u(x, t)$ can be extended as a solution of the heat equation into a region $-\rho < t \leq 0$. See D. V. Widder [1962, Corollary 3.1b].

We can show also that the nondecreasing character of $\alpha(y)$ is an essential hypothesis in Theorem 1.

Example C.

$$\alpha(y) = 1, \quad |t| < 1,$$

$$= 0, \quad |t| > 1,$$

$$u(x, t) = k(x + 1, t) - k(x - 1, t).$$

The function $u(x, t)$ is singular at $t=0$ for every $x$ except $x=0$, when $u(0, t) \equiv 0$. Here $\alpha(y)$ is not absolutely continuous or monotonic.

References


