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TIME-VARIABLE SINGULARITIES FOR SOLUTIONS OF THE HEAT EQUATION

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ABSTRACT. A solution $u(x,t)$ of the two-dimensional heat equation $u_{xx}=u_t$ may have the representation

$$u(x,t)=\int_{-\infty}^{\infty} k(x-y,t)\,d\alpha(y)$$

where $k(x,t)=(4\pi t)^{-1/2}\exp[-x^2/(4t)]$, valid in some strip $0<t<c$ of the x, t-plane. If so, $u(x_0,t)$ is known to be an analytic function of the complex variable t in the disc $\operatorname{Re}(1/t)>1/c$, for each fixed real x_0. It is shown here that if $\alpha(y)$ is nondecreasing and not absolutely continuous then $u(x_0,t)$ must have a singularity at $t=0$. Examples show that both restrictions on $\alpha(y)$ are necessary for that conclusion. It is shown further under the same hypothesis on $\alpha(y)$, that for each fixed positive $t_0<c$, $u(x,t_0)$ is an entire function of x of order 2 and of type $1/(4t_0)$. Compare the function $k(x,t)$ itself for a check on both conclusions.

1. **Introduction.** In a recent note **[1969]** H. Pollard and I discussed the representation of a function $f(x)$ by the Gaussian integral

$$f(x)=\int_{-\infty}^{\infty} k(x-y,a)\,d\alpha_a(y), \tag{1.1}$$

where $\alpha_a(y)$ is a distribution function and $k(x,t)$ is the fundamental solution of the heat equation,

$$k(x,t)=(4\pi t)^{-1/2}\exp[-x^2/(4t)].$$

We showed that if $f(x)$ has one such representation for a given constant a, then there is always a finite maximum value $a=b$ for which (1.1) holds. In fact, Pollard in an earlier note (unpublished) gave a precise formula for b in the special case in which the corresponding distribution function $\alpha_b(y)$ is not absolutely continuous. It was

$$1/b=\underset{-\infty<x<\infty}{\text{l.u.b.}}\ \limsup_{n\to\infty}\,(e/n)\,|f^{(2n)}(x)|^{1/n}.$$

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The present note evolved from my effort to simplify this formula. I show that the symbol l.u.b. may be omitted because what follows it is independent of x. I prove in fact that $f(x)$, as defined by (1.1) with $\alpha_a(y)$ a nonabsolutely continuous distribution function, is entire of order 2 and of type $1/(4a)$. Since order and type are independent of origin we have from a familiar formula (R. P. Boas [**1954**, p. 11])

$$\limsup_{n\to\infty} (1/n)\, |f^{(n)}(x)|^{2/n} = 2/(4ae).$$

We show that this equation remains true for the present function $f(x)$ when n is replaced by $2n$ and thus obtain the desired simplification of Pollard's formula.

The above conclusion about the order and type of $f(x)$ really comes as a corollary to the principal theorem of this note, one which seems to have independent significance in the theory of heat conduction. I prove that if $u(x, t)$ is a solution of the heat equation

$$\partial^2 u/\partial x^2 = \partial u/\partial t \tag{1.2}$$

which has the representation

$$u(x, t) = \int_{-\infty}^{\infty} k(x - y, t)\, d\alpha(y),$$

where $\alpha(y)$ is a nondecreasing and nonabsolutely continuous function, then $u(x, t)$, considered as a function of the complex variable t, has a singularity at $t=0$ for every real x. A case in point is $k(x, t)$ itself, for which $\alpha(y)$ is a step-function with a single positive jump. The exponential factor in $k(x, t)$ has an essential singularity at $t=0$ for every x except $x=0$, and even then the other factor has the predicted singularity. For the validity of the theorem, the hypothesis that $\alpha(y)$ is not absolutely continuous is essential. The simplest examples show this. Indeed if $\alpha(y)=y$ then $u(x, t)$ is identically equal to unity. In §4 appear further examples to show this point and another to show that the nondecreasing character of $\alpha(y)$ cannot be omitted in the hypothesis either.

2. **Time-variable singularities of temperature functions.** We now state our principal result as a theorem.

THEOREM 1. *If*

$$u(x, t) = \int_{-\infty}^{\infty} k(x - y, t)\, d\alpha(y), \qquad 0 < t < c, \tag{2.1}$$

where $\alpha(y)$ is nondecreasing and not absolutely continuous, then for every real x_0 the function $u(x_0, t)$ is analytic in the disc $\mathrm{Re}(1/t) > 1/c$ *of the complex t-plane and has a singularity at $t=0$.*

Since

$$u(x - x_0, t) = \int_{-\infty}^{\infty} k(x - y, t)\, d\alpha(y - x_0),$$

it is only necessary to consider the case $x_0 = 0$ in (2.1). That equation is then equivalent to

$$(4\pi t)^{1/2} u(0, t) = \int_0^{\infty} \exp[-y^2/(4t)]\, d[\alpha(y) - \alpha(-y)].$$

Since this integral becomes a Laplace transform in the variable $1/(4t)$ after y^2 is replaced by a new variable of integration, the stated analyticity of $u(0, t)$ is apparent. The disc in question has its center at the real point $t = c/2$ of the complex t-plane and has radius $c/2$.

To prove that $t = 0$ is a singularity we assume the contrary and seek a contradiction. That is, we assume

$$u(0, t) = \sum_{n=0}^{\infty} a_n t^n, \qquad |t| < \rho. \tag{2.2}$$

I have proved elsewhere (D. V. Widder [**1970**]) that any such function has the integral representation

$$u(0, t) = \int_0^{\infty} k(y, t) \varphi(y)\, dy, \tag{2.3}$$

where $\varphi(y)$ is an even entire function. We sketch the proof, which depends on the identity

$$(2n)!\, t^n = 2(n!) \int_0^{\infty} k(y, t) y^{2n}\, dy, \qquad t > 0. \tag{2.4}$$

Combining (2.4) and (2.2) we obtain (2.3) if

$$\varphi(y) = 2 \sum_{n=0}^{\infty} n!\, a_n \frac{y^{2n}}{(2n)!}.$$

The term-by-term integration is justified by use of the relation

$$\limsup |a_n|^{1/n} \leqq 1/\rho,$$

assumed in (2.2). This relation also shows that $\varphi(y)$ is entire.

The two integral representations for $u(0, t)$ and the uniqueness theorem for Laplace transforms now insure that

$$\alpha(y) - \alpha(-y) = \int_0^{y} \varphi(r)\, dr \tag{2.5}$$

for $y \geqq 0$. Since both sides of this equation clearly represent odd functions it must also hold for $-\infty < y < \infty$. From (2.5) we have for any $h > 0$ and any y_0 that

$$[\alpha(y_0+h)-\alpha(y_0)]+[\alpha(-y_0)-\alpha(-y_0-h)]=\int_{y_0}^{y_0+h}\varphi(r)\,dr.$$

Since $\alpha(y)$ is nondecreasing it follows that

$$0\leqq\alpha(y_0+h)-\alpha(y_0)\leqq\int_{y_0}^{y_0+h}\varphi(r)\,dr,$$

$$0\leqq\alpha(-y_0)-\alpha(-y_0-h)\leqq\int_{y_0}^{y_0+h}\varphi(r)\,dr.$$

Allowing h to approach zero we see that $\alpha(y_0+)=\alpha(y_0)$ and $\alpha(-y_0)=\alpha(-y_0-)$. Since y_0 is arbitrary it is seen that $\alpha(y)$ is continuous for $-\infty<y<\infty$. Moreover,

$$0\leqq\frac{\alpha(y_0+h)-\alpha(y_0)}{h}\leqq\frac{1}{h}\int_{y_0}^{y_0+h}\varphi(r)\,dr,\qquad 0\leqq\bar{D}_+\alpha(y_0)\leqq\varphi(y_0).$$

That is, one of the Dini derivatives is finite at every point. This fact with the continuity of $\alpha(y)$ guarantees that $\alpha(y)$ is absolutely continuous. See, for example, S. Saks [**1937**, Theorem 4.6 on p. 271 and Theorem 6.7 on p. 227]. But this contradicts a hypothesis of the theorem, as desired.

3. **Consequences of the main theorem.** The previous result has immediate consequences for functions $f(x)$ which have the integral representation (1.1).

THEOREM 2. *If*

$$f(x)=\int_{-\infty}^{\infty}k(x-y,c)\,d\alpha(y),$$

where $\alpha(y)$ *is nondecreasing, bounded and not absolutely continuous, then for all real* x *and for* $|t|<c$,

$$\sum_{n=0}^{\infty}\frac{(-t)^n}{n!}f^{(2n)}(x)=\int_{-\infty}^{\infty}k(x-y,c-t)\,d\alpha(y). \tag{3.1}$$

The function

$$u(x,t)=\int_{-\infty}^{\infty}k(x-y,t)\,d\alpha(y) \tag{3.2}$$

satisfies the hypotheses of Theorem 1 for every positive c since the integral (3.2) clearly converges for all positive t. Hence by that theorem $u(x_0,t)$ is analytic in the complex half-plane $\operatorname{Re} t>0$. Consequently the Taylor expansion

$$u(x_0,c-t)=\sum_{n=0}^{\infty}\frac{\partial^n u(x_0,c)}{\partial t^n}\frac{(-t)^n}{n!} \tag{3.3}$$

is valid for $|t|<c$. But since $u(x, t)$ satisfies the heat equation (1.2) we see that

$$\frac{\partial^n}{\partial t^n} u(x_0, c) = \frac{\partial^{2n}}{\partial x^{2n}} u(x_0, c) = f^{(2n)}(x_0),$$

and equation (3.3) is equivalent to (3.1).

Corollary 2.1. *If $f(x)$ is defined as in Theorem 2, then*

$$\limsup_{n\to\infty} \frac{|f^{(2n)}(x)|^{1/n}}{n} = \frac{1}{ce}, \qquad -\infty < x < \infty. \tag{3.4}$$

For, by Theorem 1, the series (3.1) defines a function of t which has a singularity at $t=c$. Hence its radius of convergence is c for every real x. That is,

$$\limsup_{n\to\infty} \left[\frac{|f^{(2n)}(x)|}{n!}\right]^{1/n} = \frac{1}{c}.$$

Stirling's formula now yields (3.4).

Corollary 2.2. *If $f(x)$ is defined as in Theorem 2, then*

$$\limsup_{n\to\infty}(1/n)\, |f^{(n)}(x)|^{2/n} = 1/(2ce). \tag{3.5}$$

so that $f(x)$ is entire of order 2 and of type $1/(4c)$.

By familiar properties of the Weierstrass transform we have from equation (3.2) that

$$\frac{\partial}{\partial x} u(x, t) = \int_{-\infty}^{\infty} \frac{\partial}{\partial x} k(x - y, t)\, d\alpha(y), \tag{3.6}$$

valid for $\operatorname{Re} t>0$. As before, the Taylor expansion

$$\frac{\partial}{\partial x} u(x_0, c - t) = \sum_{n=0}^{\infty} \frac{\partial^{n+1}}{\partial x\, \partial t^n} u(x_0, c) \frac{(-t)^n}{n!} = \sum_{n=0}^{\infty} f^{(2n+1)}(x_0) \frac{(-t)^n}{n!}$$

must hold at last for $|t|<c$. Hence

$$\limsup_{n\to\infty} \left|\frac{f^{(2n+1)}(x_0)}{n!}\right|^{1/n} \leqq \frac{1}{c}.$$

This is equivalent to

$$\limsup_{n\to\infty} 1/(2n + 1)\, |f^{(2n+1)}(x_0)|^{2/(2n+1)} \leqq 1/(2ce).$$

This inequality combined with equation (3.4) yields equation (3.5), and the proof is complete. Unlike the function $u(x, t)$ of equation (3.2) its derivative (3.6) need not be singular at $t=0$ for all x. Indeed if $\alpha(y)$ is odd, the derivative is identically zero when $x=0$.

4. **Examples.** In the introduction we observed that for the validity of Theorem 1 it is essential that $\alpha(y)$ should not be absolutely continuous.

We give further examples here to show this.

EXAMPLE A.

$$u(x, t) = \int_0^\infty k(x-y, t)\, dy = \tfrac{1}{2}\operatorname{erfc}(-x/(4t)^{1/2}).$$

Here $\alpha(y)$ is nondecreasing and abolutely continuous. The function $u(x, t)$ is singular at $t=0$ for every x except $x=0$ when $u(0, t)\equiv\frac{1}{2}$.

EXAMPLE B.

$$u(x, t) = \int_{-\infty}^\infty k(x-y, t)\exp[-y^2]\, dy = \frac{\exp[-x^2/(1+4t)]}{(1+4t)^{1/2}}.$$

Again $\alpha(y)$ is nondecreasing and absolutely continuous, even bounded. The function $u(x, t)$ is not singular at $t=0$ for any x. Note that this type of situation always obtains when $\alpha(y)$ is entire of order 2 and of finite type. For then $u(x, t)$ can be extended as a solution of the heat equation into a region $-\rho<t\leqq 0$. See D. V. Widder [**1962**, Corollary 3.1b].

We can show also that the nondecreasing character of $\alpha(y)$ is an essential hypothesis in Theorem 1.

EXAMPLE C.

$$\begin{aligned} \alpha(y) &= 1, \qquad |t| < 1, \\ &= 0, \qquad |t| > 1, \\ u(x, t) &= k(x+1, t) - k(x-1, t). \end{aligned}$$

The function $u(x, t)$ is singular at $t=0$ for every x except $x=0$, when $u(0, t)\equiv 0$. Here $\alpha(y)$ is not absolutely continuous or monotonic.

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