STRONGLY DISSIPATIVE OPERATORS AND NONLINEAR EQUATIONS. IN A FRÉCHET SPACE

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Abstract. Suppose that $X$ is a Fréchet space, $Y$ is a Banach subspace of $X$, and $A$ is a function from $Y$ into $X$. Sufficient conditions are determined to insure that the equation $Ax = y$ ($y \in Y$) has a unique solution $x_y$ which depends continuously on $y$. The techniques of this paper use the theory of dissipative operators in a Banach space, and the results are associated with the idea of admissibility of the space $Y$. Also, the equation $Ax = Cx + y$ is considered where $C$ is completely continuous.

Let $X$ be a Fréchet space over the real or complex field (i.e., $X$ is a locally convex, complete, metrizable topological vector space—see e.g., [10, p. 85]). In this paper we assume the following:

(X1) $(q_n)_n$ is an increasing family of continuous seminorms on $X$ which defines the topology of $X$ (i.e., $q_n \leq q_{n+1}$ and $X - \lim_{k \to \infty} x_k = x$ if and only if $\lim_{k \to \infty} q_n(x_k - x) = 0$ for each $n$).

(X2) $Y = \{x \in X : \sup_n \{q_n(x)\} < \infty\}$ and $|x| = \sup_x \{q_n(x)\}$ for each $x$ in $Y$.

Note that a sequence of seminorms satisfying (X1) always exists, and the space $Y$ in (X2) is a complete normed space with norm $|\cdot|$. However, the members of $Y$ depend on how the sequence $(q_n)_n$ is chosen.

In this paper, some results on strongly dissipative operators in a Banach space are used to establish analogous results for a class of operators which map $Y$ into $X$. Recently, some fixed point theorems for completely continuous perturbations of Lipschitz continuous functions in locally convex spaces have been obtained by Cain and Nashed [1]. In this paper, a class of functions $A$ from $Y$ into $X$ is considered which can be "approximated" by functions which are defined on $X$. Sufficient conditions are established to insure that the equation $Ax = y$ has a unique solution $x_y$ for each $y$ in $Y$, and the function $B$ defined by $By = x_y$ for each $y$ in $Y$ has certain continuity properties. This result is closely associated with the notion of admissibility introduced by Massera and Schäffer [6]. Recently, admissibility has been used in studying existence and stability of solutions to Volterra integral equations—see Corduneanu [2], [3] and Miller [8].

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We also consider the solvability of the equation $Ax = Cx + y$ where $C$ is a function from $Y$ into $Y$ which has certain compact properties and growth properties. Finally, two examples of integral equations on the half-line are given to illustrate when these techniques may be applied.

**Definition 1.** If $A$ is a function from $X$ into $X$, then $A$ is said to be compatible with $q_n$ if whenever $x$ is in $X$ and $(x_k)_1^n$ is a sequence in $X$ such that $\lim_{k \to \infty} q_n(x_k - x) = 0$, it follows that $\lim_{k \to \infty} q_n(Ax_k - Ax) = 0$.

For each positive integer $n$ let $M_n = \{x \in X : q_n(x) = 0\}$ and let $X/M_n = \{\phi(x) : x \in X \text{ and } \phi(x) = x + M_n\}$. If $q_n^*(\phi(x)) = q_n(x)$ for each $\phi(x)$ in $X/M_n$, then $q_n^*$ is well defined and is a norm on the quotient space $X/M_n$. If $A$ is a function from $X$ into $X$ which is compatible with $q_n$, then define $A^*(\phi(x)) = \phi(Ax)$ for each $\phi(x)$ in $X/M_n$. It is easy to see that $A^*$ is well defined and continuous on $X/M_n$.

**Definition 2.** Suppose that $X/M_n$, $A$ and $A^*$ are as in the above paragraph and let $E_n$ denote the completion of the normed space $X/M_n$ with the norm on $E_n$ denoted by $q_n'$. Then $A$ is said to be strongly dissipative with respect to $q_n$ if there is a continuous function $\alpha_n$ from $[0, \infty)$ into $[0, \infty)$ and a continuous function $A'$ from $E_n$ into $E_n$ such that

$$\alpha_n(0) = 0, \quad \alpha_n(s) > 0 \text{ if } s > 0,$$

$$A^*(\phi(x)) = A'(\phi(x))$$

if $\phi(x)$ is in $X/M_n$, and

$$\lim_{h \to 0^+} \frac{[q_n'(x - y + h(Ax - Ay)) - q_n'(x - y)]}{h} \leq -\alpha_n(q_n'(x - y))$$

for all $x$ and $y$ in $E_n$.

**Remark.** In many cases the quotient spaces $X/M_n$ are complete—for example, the spaces considered in Examples 1 and 2 below have this property. If $X/M_n$ is complete and $A$ is compatible with $q_n$, note that $A$ is strongly dissipative with respect to $q_n$ only in case

$$\lim_{h \to 0^+} \frac{[q_n(x - y + h(Ax - Ay)) - q_n(x - y)]}{h} \leq -\alpha_n(q_n(x - y))$$

for all $x$ and $y$ in $X$. Also, if the function $A^*$ is uniformly continuous on bounded subsets of $X/M_n$, then $A^*$ can be extended to a continuous function $A'$ on $E_n$, and it can be shown that $A$ is strongly dissipative with respect to $q_n$.

**Lemma 1.** Suppose that $A$ is a function from $X$ into $X$ which is strongly dissipative with respect to $q_n$. Then, for each $z$ in $X$ and each $\varepsilon > 0$, there is a $x^*_\varepsilon$ in $X$ such that $q_n(Ax^*_\varepsilon - z) \leq \varepsilon$.

**Indication of Proof.** If the function $A'$ is as in Definition 2, it is easy to see that $A'$ satisfies each of the suppositions of Theorem 2 in [7], and so there is a unique $y'$ in $E_n$ such that $q_n'(A'y' - \phi(z)) = 0$. Since $X/M_n$ is dense in $E_n$ and $A'$ is continuous, there is an $x^*_\varepsilon$ in $X$ such that
Thus, \( q_n(Ax_z^2 - z) = q_n^*(A^* \phi(x_z^2) - \phi(z)) \leq \varepsilon \) and the lemma is true.

We now prove our main result.

**Theorem 1.** Suppose that conditions (X1)—(X2) are fulfilled, \( \alpha \) is a continuous, increasing function from \([0, \infty)\) into \([0, \infty)\) such that \( \alpha(0) = 0 \) and \( \lim_{s \to \infty} \alpha(s) = \infty \), \( A \) is a function from \( Y \) into \( X \), and \( (A_n)_{n \in \mathbb{N}} \) is a sequence of functions from \( X \) into \( X \) such that

(i) \( A_n \) is strongly dissipative with respect to \( q_n \) for each \( n \);
(ii) \( \lim_{n \to \infty} [q_n(x - y + h(A_n x - A_n y)) - q_n(x - y)] / h \leq -\alpha(q_n(x - y)) \) for each \( x \) and \( y \) in \( X \) and each \( n \);
(iii) there is a number \( L > 0 \) such that \( q_n(An_0) \leq L \) for each \( n \);
(iv) for each pair of positive numbers \( K \) and \( \delta \) there is a positive integer \( N(K, \delta) \) such that if \( j \geq N(K, \delta) \) and \( x \) is a member of \( X \) such that \( q_j(x) \leq K \), then \( q_j(Ax - A x) \leq \delta \); and
(v) if \( K \) is a positive number, then \( \lim_{i \to \infty} q_i(Ax - A x) = 0 \), uniformly for \( x \) in \( Y \) with \( |x| \leq K \).

Then, for each \( z \) in \( Y \), there is a unique \( x_z \) in \( Y \) such that \( Ax_z = z \). Furthermore, if the function \( B \) from \( Y \) into \( Y \) is defined by \( Bz = x_z \) for each \( z \) in \( Y \), then \( |Bz - Bw| \leq \alpha^{-1}(|z - w|) \) for all \( z \) and \( w \) in \( Y \).

**Remark.** Note that the function \( A \) in Theorem 1 need not be defined on all of \( X \) and does not necessarily map \( Y \) into \( Y \). Furthermore, by supposition (ii) if the quotient spaces \( X/M_n \) are complete for each \( n \), then we need only assume that \( A_n \) is compatible with \( q_n \) in supposition (i)—see remark following Definition 2.

**Proof of Theorem 1.** Note that supposition (ii) implies that

\[
q_n(x - y) \leq \alpha^{-1}(q_n(A_n x - A_n y))
\]

for all \( x \) and \( y \) in \( X \). By Lemma 1 and suppositions (i) and (ii), there is an \( x^n_z \) in \( X \) such that \( q_n(A_n x^n_z - z) \leq n^{-1} \) for each \( z \) in \( Y \) and positive integer \( n \). Since \( \alpha^{-1} \) is increasing, we have from (1) and supposition (iii) that

\[
q_n(x^n_z) \leq \alpha^{-1}(q_n(A_n x^n_z - A_n 0)) \leq \alpha^{-1}(|z| + 1 + L)
\]

for each \( n \). Let \( \varepsilon \) be a positive number and let \( \delta > 0 \) be such that \( \alpha^{-1}(s) \leq \varepsilon \) whenever \( 0 \leq s \leq 2\delta \). If \( N \) is as in (iv) and \( j \geq N(\alpha^{-1}(|z| + 1 + L), \delta) \), then \( q_j(x^n_z) \leq \alpha^{-1}(|z| + 1 + L) \), so

\[
q_j(A_n x^n_z - A_n x^n_z) \leq q_j(z - A_n x^n_z) + \alpha^{-1}
\]

\[
\leq q_j(z - A_n x^n_z) + q_j(A_n x^n_z - A_n x^n_z) + \alpha^{-1}
\]

\[
\leq 2\alpha^{-1} + q_j(A_n x^n_z - A_n x^n_z) \leq 2\alpha^{-1} + \delta.
\]
Now let \( m \) be a positive integer such that
\[
m \geq \max\{N(x^{-1}(|z|+1+L), \delta), 2/\delta\}.
\]
If \( j \geq i \geq m \), then by (1), (3), and the choice of \( \delta \),
\[
q_i(x_i^j - x_i^k) \leq x^{-1}(q_i(A_i x_i^j - A_i x_i^k)) \leq x^{-1}(2\delta) \leq \varepsilon.
\]
It now follows easily from (4) and (X1) that \((x_i^j)_i^\infty\) is a Cauchy sequence in \( X \). Since \( X \) is complete, there is an \( x_z \) in \( X \) such that \( X - \lim_{k \to \infty} x_z^k = x_z \).

By (2),
\[
q_n(x_z) = \lim_{k \to \infty} q_n(x_z^k) \leq \limsup_{k \to \infty} q_k(x_z^k) \leq x^{-1}(|z| + 1 + L)
\]
for each \( n \), so \( x_z \) is in \( Y \) with \( |x_z| \leq x^{-1}(|z| + 1 + L) \). Letting \( j \to \infty \) in the term on the left side of (3) shows that \( \lim_{i \to \infty} q_i(A_i x_i^j - A_i x_z) = 0 \). Thus, from supposition (v), if \( n \) is a positive integer,
\[
q_n(A x_z - z) \leq \lim_{i \to \infty} q_i(A x_z - z) = \lim_{i \to \infty} q_i(A_i x_i^j - z)
\]
\[
\leq \lim_{i \to \infty} q_i(A_i x_i^j - A_i x_z^j) + i^{-1} = 0.
\]
Hence \( A x_z = z \). If \( y \) is in \( Y \) and \( A y = z \), then by (1) and supposition (v),
\[
q_n(y - x_z) \leq \lim_{i \to \infty} q_i(y - x_z) \leq \lim_{i \to \infty} x^{-1}(q_i(A_i y - A_i x_z))
\]
\[
\leq \lim_{i \to \infty} x^{-1}(q_i(A_i y - z) + q_i(z - A_i x_z)) = x^{-1}(0) = 0
\]
for each \( n \). Consequently \( y = x_z \) and the function \( B \) defined in the statement of the theorem is well defined. Furthermore, if \( z \) and \( w \) are in \( Y \) and \( (x_i^j)_i^\infty \) and \( (x_i^m)_i^\infty \) are as constructed above, then by (1) and the fact that
\[
q_i(A_i x_i^j - A_i x_i^m) \leq q_i(z - w) + 2i^{-1},
\]
\[
q_n(x_z - x_m) = \lim_{i \to \infty} q_n(x_i^j - x_i^m) \leq \limsup_{i \to \infty} q_i(x_i^j - x_i^m)
\]
\[
\leq \limsup_{i \to \infty} x^{-1}(q_i(A_i x_i^j - A_i x_i^m))
\]
\[
\leq \limsup_{i \to \infty} x^{-1}(q_i(z - w) + 2i^{-1}) \leq x^{-1}(|z - w|).
\]
Thus,
\[
|B z - B w| = \sup\{q_n(x_z - x_m) : n = 1, 2, \cdots\} \leq x^{-1}(|z - w|)
\]
and the proof of Theorem 1 is complete.

**Lemma 2.** Let the suppositions of Theorem 1 be fulfilled and for each \( R > 0 \) let \( Q_R = \{x \in Y : |x| \leq R\} \). Then, considering \( Q_R \) with the topology induced by \( X \), the function \( B \) defined in Theorem 1 is continuous from \( Q_R \) into \( X \).
Indication of Proof. Let $x$ be in $Q_R$ and let $(x_k)_k$ be a sequence in $Q_R$ such that $\lim_{k \to \infty} q_n(x_k - x) = 0$ for each $n$. Let $n$ be a positive integer, let $\varepsilon > 0$, and let $\delta > 0$ be such that $\alpha^{-1}(s) \leq \varepsilon$ whenever $0 \leq s \leq \delta$. Note that if $y$ is in $Q_R$, then $|By| \leq |By - BO| + |BO| \leq \alpha^{-1}(R) + |BO|$. By supposition (v) of Theorem 1, let the integer $m \geq n$ be such that $q_m(A_m y - Ay) \leq \delta/3$ for all $y$ in $Y$ with $|y| \leq \alpha^{-1}(R) + |BO|$. Let $p$ be a positive integer such that $q_m(x_k - x) \leq \delta/3$ whenever $k \geq p$. If $k \geq p$, we have from (1), the choice of $\delta$, and the fact that $\alpha^{-1}$ is increasing, that

$$q_n(Bx_k - Bx) \leq q_m(Bx_k - Bx) \leq \alpha^{-1}(q_m(A_m Bx_k - A_m Bx))$$

$$\leq \alpha^{-1}(q_m(A_m Bx_k - ABx_k) + q_m(ABx_k - ABx)) + q_m(ABx - A_m Bx))$$

$$\leq \alpha^{-1}(\delta/3 + q_m(x_k - x) + \delta/3) \leq \alpha^{-1}(\delta) \leq \varepsilon.$$

Thus $\lim_{k \to \infty} q_n(Bx_k - Bx) = 0$ for each $n$, and the assertion of the lemma follows.

Theorem 2. Let the suppositions of Theorem 1 be fulfilled and suppose that $C$ is a function from $Y$ into $Y$ such that

$$\limsup_{|x| \to \infty} \frac{|C x|}{|x|} = \beta < 1.$$

Suppose further that at least one of the following is satisfied:

(i) $C$ is continuous from the Banach space $Y$ into itself and maps bounded subsets of $Y$ into relatively compact subsets of $Y$; or

(ii) if $Q_R$ is as in Lemma 2 (with the topology induced by $X$), then, for each $R > 0$, $C$ is continuous from $Q_R$ into $X$ and the image of $Q_R$ under $C$ is relatively compact in $X$.

Then, for each $z$ in $Y$, there is a $y_z$ in $Y$ such that $A y_z - C y_z = z$.

Indication of Proof. Note that we need only show that there is an $x_0$ in $Y$ such that $A x_0 - C x_0 = 0$, or equivalently, $B - C x_0 = x_0$ where $B$ is as defined in Theorem 1. Since $\limsup_{|x| \to \infty} \frac{|B - C x|}{|x|} < 1$, let $r_1 > 0$ be such that $|B - C x| \leq |x|$ whenever $|x| \geq r_1$ and let $r_2 = \sup \{|B - C x| : |x| \leq r_1\}$. If $R = \max(r_1, r_2)$, then $B - C$ maps $Q_R$ into $Q_R$. If (i) holds the theorem follows from the Schauder fixed point theorem, and if (ii) holds the theorem follows from Lemma 2 and the Tychonov fixed point theorem (see e.g. [5, Theorem 5, p. 456]).

Example 1. Let $X$ be the space $L^2[0, \infty)$ of all measurable functions $x$ from $[0, \infty)$ into the real numbers such that $q_n(x) = (\int_0^\infty |x(s)|^2 \, ds)^{1/2} < \infty$ for each positive integer $n$. Note that $Y$ is the space $L^2[0, \infty)$ and $|x| = (\int_0^\infty |x(s)|^2 \, ds)^{1/2}$ for each $x$ in $Y$. Let $a$ and $b$ be symmetric, measurable, locally integrable functions defined on $[0, \infty)^2$ such that the operator
$Tx(t)=\int_0^t a(t, s)x(s)\,ds$ maps $L^2[0, \infty)$ continuously into $L^2[0, \infty)$ and the operator $Sx(t)=\int_0^\infty b(t, s)x(s)\,ds$ maps $L^2[0, \infty)$ into $L^2[0, \infty)$. Assume further that (a) $\int_0^\infty a(t, s)x(s)x(t)\,ds\,dt \geq 0$ for each $x$ in $L^2[0, \infty)$ and each $n$; (b) $\int_0^\infty b(t, s)x(s)x(t)\,ds\,dt \geq 0$ for each $x$ in $L^2[0, \infty)$ and each $n$; and (c) $\int_0^\infty (|b(t, s)|^2)\,ds\,dt < \infty$. Define $Ax=-x-Tx-Sx$ for each $x$ in $L^2[0, \infty)$. Then the suppositions of Theorem 1 are fulfilled with $\alpha(s)=s$ for all $s \geq 0$ and $A_nx=-x-Tx-S_nx$ where $S_nx(t)=\int_0^\infty b(t, s)x(s)\,ds$ for each $x$ in $L^2[0, \infty)$. Suppositions (i) and (iii) of Theorem 1 are easily seen to be true. Supposition (ii) is immediate from (a) and (b) above and the fact that

$$2\int_0^n \left[ \int_0^t a(t, s)x(s)\,ds \right] x(t)\,dt = \int_0^n \int_0^n a(t, s)x(s)x(t)\,ds\,dt,$$

which follows from the symmetry of $a$. Since condition (c) above implies $\lim_{n \to \infty} \int_0^\infty |b(t, s)|^2\,ds\,dt = 0$, it is easy to see that suppositions (iv) and (v) are fulfilled. Thus, for each $z$ in $L^2[0, \infty)$, there is a unique $x_z$ in $L^2[0, \infty)$ such that

$$x_z(t) + \int_0^t a(t, s)x_z(s)\,ds + \int_0^\infty b(t, s)x_z(s)\,ds = z(t)$$

for almost all $t$ in $[0, \infty)$. Also, $\int_0^\infty |x_z(s)|^2\,ds \leq \int_0^\infty |z(s)|^2\,ds$. Note the operator defined by the left side of (5) is not necessarily defined on all of $L^2[0, \infty)$ and need not map $L^2[0, \infty)$ into itself.

**Example 2.** Let $C_c[0, \infty)$ denote the Fréchet space of all continuous functions $x$ from $[0, \infty)$ into the $m$ dimensional space $R^m$ (with $\|\cdot\|$ denoting an appropriate norm on $R^m$) with the topology generated by uniform convergence on compact subsets of $[0, \infty)$. Define $q_n(x)=\max\{\|x(t)\|:0 \leq t \leq n\}$ for each $x$ in $C_c[0, \infty)$ and each positive integer $n$. Then $Y$ is the space $BC[0, \infty)$ of all $x$ in $C_c[0, \infty)$ such that $|x|:=\sup\{\|x(t)\|:t \geq 0\} < \infty$. Now let $f$ and $g$ be continuous functions from $[0, \infty)^2 \times R^m$ into $R^m$ such that (a) $f(t, s, 0)=g(t, s, 0)=0$ for all $(t, s)$ in $[0, \infty)^2$; (b) $f(t, s, \xi)=0$ for all $(t, s, \xi)$ in $[0, \infty)^2 \times R^m$ with $s \geq t$; and there are continuous functions $\theta$ and $\phi$ from $[0, \infty)$ into $[0, \infty)$ such that (c) $\sup\{\|x(t)\|:t \geq 0\} = \lambda < 1$; (d) $\sup\{\|\phi(t, s)\|:t \geq 0\} = \gamma < 1 - \lambda$; (e) $\|f(t, s, \xi)-f(t, s, \xi_2)\| \leq \theta(t, s)\|\xi_1-\xi_2\|$ for all $(t, s, \xi_1)$ and $(t, s, \xi_2)$ in $[0, \infty)^2 \times R^m$; and (f) $\|g(t, s, \xi)\| \leq \phi(t, s)\|\xi\|$ for all $(t, s, \xi)$ in $[0, \infty)^2 \times R^m$. If $S$ is the integral operator defined on $C_c[0, \infty)$ by $Sx(t)=\int_0^t f(t, s, x(s))\,ds$, then the suppositions of Theorem 1 are easily seen to be fulfilled with $A=-x+Sx$ for each $x$ in $BC[0, \infty)$, $A_nx=-x+S_nx$ for each $x$ in $C_c[0, \infty)$, and $\alpha(s)=(1-\lambda)s$ for each $s \geq 0$. Let $T$ be the integral operator defined in $BC[0, \infty)$ by $Tx(t)=\int_0^t g(t, s, x(s))\,ds$ and suppose that $T$ maps $BC[0, \infty)$ into $BC[0, \infty)$. Note that conditions (d) and (f)
above imply that $|Tx| \leq \gamma|x|$ for all $x$ in $BC[0, \infty)$. Now let $z$ be in $BC[0, \infty)$ and let $Cx = Tx + z$ for each $x$ in $BC[0, \infty)$. If $B$ is as in Theorem 1 (i.e., $A \cdot Bx = x$ for all $x$ in $BC[0, \infty)$), then $BO = 0$ and we have that $|Bx| \leq |(1-\lambda)^{-1}|x|$ for all $x$ in $BC[0, \infty)$. It now follows easily that the operator $B \cdot C$ maps the ball $Q = \{x \in BC[0, \infty) : |x| \leq |z|(1-\gamma-\lambda)^{-1}\}$ into itself. Thus, if $C$ is completely continuous for the $BC[0, \infty)$ topology on $Q$ or if $C$ is completely continuous for the $C_c[0, \infty)$ topology on $Q$ (note that this is the case if
\[
\lim_{p \to \infty} \sup_{t \leq p} \left( \int_0^t \phi(t, s) \, ds : 0 \leq t \leq p \right) = 0,
\]
where $\phi$ is as in (d)), then there is an $x \in BC[0, \infty)$ with $|x| \leq |z|(1-\gamma-\lambda)^{-1}$ such that
\[
(x(t) - \int_0^t f(t, s, x(s)) \, ds = \int_0^\infty g(t, s, x(s)) \, ds + z(t)
\]
for all $t$ in $[0, \infty)$. Furthermore, since $|B \cdot Cx| < |x|$ if $|x| > |z|(1-\gamma-\lambda)^{-1}$, all solutions $x_\lambda$ to (6) satisfy $|x_\lambda| \leq |z|(1-\gamma-\lambda)^{-1}$. In particular $|x_\lambda| \to 0$ as $|z| \to 0$, so we have a type of stability criteria for the zero solution of equation (6) when $z(t) = 0$ for all $t \geq 0$.

Remark. In Example 2, we need only assume that the inequalities (e) and (f) hold in some neighborhood of the origin in $R^m$. This follows from the fact that if $r > 0$ and $h$ is a function from $Q_r = \{x \in R^m : ||x|| \leq r\}$ into $R^m$ such that $\|h(\xi_1) - h(\xi_2)\| \leq k\|\xi_1 - \xi_2\|$ for all $\xi_1$ and $\xi_2$ in $Q_r$, then there is a function $h^*$ from $R^m$ into $R^m$ such that $h^*(\xi) = h(\xi)$ for all $\xi$ in $Q_r$ and $\|h^*(\xi_1) - h^*(\xi_2)\| \leq k^*\|\xi_1 - \xi_2\|$ for all $\xi_1$ and $\xi_2$ in $R^m$ where $k^* = k$ if $||\cdot||$ is the Euclidean norm on $R^m$. Thus the stability and existence criteria established in Example 2 give some improvements to those of Miller, Nohel and Wong [9].

References

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