NOTE RELATING BOCHNER INTEGRALS
AND REPRODUCING KERNELS TO SERIES
EXPANSIONS ON A GAUSSIAN BANACH
SPACE

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Abstract. Fernique's recent proof of finiteness of positive
moments of the norm of a Banach-valued Gaussian random vector
$\mathcal{X}$ is used to prove $r$th mean convergence of reproducing kernel
series representations of $\mathcal{X}$. Embedding of the reproducing kernel
Hilbert space into the Banach range of $\mathcal{X}$ is explicitly given by
Bochner integration. This work extends and clarifies work of
Kuelbs, Jain and Kallianpur.

Fernique [2] has recently proved in a most elementary way that
\[
\lim_{n \to +} E \exp \alpha \|X\|^2_n < \infty
\]
for every centered Gaussian random vector $X$ taking values in a real and separable Banach space $B$. As will be shown
below, this result can be used to provide a dramatically simple proof of
the strong convergence of certain representations of $\mathcal{X}$ by a series in $B$, as
given by Kuelbs [4] and Jain-Kallianpur [3]. The role of reproducing
kernel Hilbert spaces in such representations is sharply revealed by this
approach.

In this paper, $B$ is a real and separable Banach space, $B^*$ its topological
dual, $\mathcal{B}$ is the $\sigma$-algebra generated by the open subsets of $B$, and $P$ is a
probability measure on $\mathcal{B}$ for which the induced distributions of the
random variables $x^* \in B^*$ are all Gaussian with zero means.

Suppose that in addition to being a Banach space, $B$ is also a subset of
the set of real functions on a set $T$ (distinct points of $B$ also being distinct
as real functions on $T$), and that for each $t \in T$ the evaluation mapping $X_t$
defined by $X_t(x) = x(t)$, $x \in B$, is continuous on $B$. For example, if $T$
is taken equal to $B^*$, each $x \in B$ may be viewed as the continuous linear
evaluation function on $B^*$ defined by $x(x^*) = x^*(x)$, $x^* \in B^*$. Let $L$
denote $P$ quadratic-mean closure of $\{X_t, t \in T\}$ viewed as a Hilbert subspace of

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$L_2(B, \mathcal{B}, P)$, and denote by $H(R)$ the reproducing kernel Hilbert space of

$$R(s, t) = \int_B x(s)x(t)P(dx), \quad (s, t) \in T \times T.$$  

It is well known (e.g. see [5]) that $\mathcal{L}$ is isometrically isomorphic to $H(R)$ under the linear extension of the mapping $X_t \mapsto R(t, \cdot)$, $t \in T$, and that $H(R)$ is characterized as the unique Hilbert space of real functions on $T$ containing the sections $R(t, \cdot)$, $t \in T$, and satisfying $f(t) = (f, R(t, \cdot))_{H(R)}$ for every $t \in T$, $f \in H(R)$. The latter is termed the reproducing property and this isomorphism of $\mathcal{L}$ with $H(R)$ is termed the natural isomorphism.

**Proposition.** Consequent to the preceding assumptions:

(a) If $L \in \mathcal{L}$, the element of $H(R)$ to which $L$ corresponds under the natural isomorphism is given by the convergent Bochner integral

$$x_L = \left. \int_B L(x)xP(dx) \right|_{H(R)} \in B$$

which may be calculated pointwise

$$x_L(t) = \int_B L(x)x(t)P(dx), \quad t \in T.$$

(b) If $L_1, L_2, \ldots$ are a complete orthonormal set for $\mathcal{L}$, then for every $r \geq 1$, as $n \to \infty$,

$$\int_B \left\| x - \sum_{k=1}^n L_k(x)L_k \right\|_B^r P(dx) \to 0.$$  

In particular, this is a series representation by $H(R)$ functions, the series converges almost surely, and closure of $H(R)$ in $B$ gives the support of $P$.

**Proof.** Suppose $L \in \mathcal{L}$. Let $v(x) = L(x)x$, $x \in B$. Then $v$ is a Banach-valued random vector and for every $r \geq 1$,

$$\int_B \|v(x)\|_B^r P(dx) \leq \left( \int_B L^2r(x)P(dx) \int_B \|x\|_B^{2r} P(dx) \right)^{1/2}. $$

Since $L$ is the $P$ quadratic-mean limit of $P$-Gaussian random variables, $L$ is itself $P$-Gaussian and $\int_B L^2r_B(x)P(dx) < \infty$. By Fernique's result quoted earlier, $\int_B \|x\|_B^{2r}P(dx) < \infty$. Therefore $\int_B \|v(x)\|_B^r P(dx) < \infty$ for each $r \geq 1$.

(a) Suppose $L \in \mathcal{L}$. Taking $r=1$ in the above we conclude [1] that the Bochner integral $x_L = \int_B L(x)xP(dx) \in B$ exists. For each $t \in T$, continuity

\footnote{In fact $\|x_L\|_B \leq \|x_L\|_{H(R)}\|P\|$ where $\|P\|^2 = \int_B \|x\|_B^2 P(dx) < \infty$, follows immediately once it is established that $x_L$ is companion to $L$ under the natural isomorphism.}
of the linear \( X_t \) enables passage of \( X_t \) inside the Bochner integral. Let \( \hat{L} \in H(R) \) correspond to \( L \) under the natural isomorphism. Then, for each \( t \in T \),

\[
\hat{L}(t) = (\hat{L}, R(t, \cdot ))_{H(H)} = (L, X_t) = x_L(t).
\]

(b) Suppose \( L_1, L_2, \ldots \) are a complete orthonormal set for \( L \). These \( L_1, L_2, \ldots \) have the \( P \)-law of independent and identically distributed Gaussian random variables with means zero and variances unity. If \( n > 0 \), \( \hat{L} \in (L_1, \ldots, L_n)_x \) (the submanifold of \( L \) spanned by \( L_1, \ldots, L_n \)) then

\[
\sum_{k=1}^{k=n} L_k x_k \Leftrightarrow \sum_{k=1}^{k=n} L_k L(x_k) P(dx) = L \quad \text{a.e. } P.
\]

Therefore

\[
\sum_{k=1}^{k=n} L_k x_k, \quad \mathcal{F}_n = \sigma \{L_1, \ldots, L_n\}, \quad n \geq 1,
\]

is a strong martingale in the sense of [1]. That is, for \( n \geq 1 \),

\[
(A \in \mathcal{F}_n) \Rightarrow \left( \int \sum_{k=1}^{k=n} L_k(x) x_k P(dx) = \int x P(dx) \right).
\]

Since \( B \subseteq P \)-completion of \( \sigma\{U_n, \mathcal{F}_n\} \), we conclude from [1, Theorem 1] that, as \( n \to \infty \),

\[
\left\| \hat{x} - \sum_{k=1}^{k=n} L_k(x) x_k \right\|_{H^1} \to 0 \quad \text{a.e. } P.
\]

This implies that closure of \( H(R) \) in \( B \) gives the support of \( P \). For if \( x \in B \) and every \( B \) open neighborhood of \( x \) has positive probability then there are sums of the type \( \sum_{k=1}^{k=n} L_k(x_k) x_{L_k} \) (for \( n \geq 1 \), and \( x \in B \)) of arbitrary \( B \)-closeness to \( x \), and (by (a)) belonging to \( H(R) \). If, on the other hand, there is an \( \epsilon > 0 \) and \( L \in L \) for which an \( \epsilon \)-radius \( B \)-sphere containing \( x \) has \( P \)-probability zero, it follows from mutual absolute continuity of Gaussian measures under translation by \( H(R) \) functions (e.g. see [5]) that an \( \epsilon \)-radius \( B \)-sphere containing the origin of \( B \) has zero \( P \)-probability. For the purpose of proving this impossible we may as well assume this sphere is centered at the origin. Then choose \( n \) sufficiently large so that with positive \( P \)-probability \( \| \hat{x} \|_{H^1} - \| \sum_{k=1}^{k=n} L_k(x) x_{L_k} \|_B < \epsilon \). Since the latter event involves only the tail of this series in mutually independent summands, it suffices to prove that \( \| \sum_{k=1}^{k=n} L_k(x) x_{L_k} \|_B \) has positive probability of being in every interval about zero. By footnote 1 however

\[
\left\| \sum_{k=1}^{k=n} L_k(x) x_{L_k} \right\|_B^2 \leq \| P \|_2 \sum_{k=1}^{k=n} L_k^2(x) < \infty
\]
and the $P$-probability that $\sum_{k=1}^{n} L_{n}(x)^{2} < \delta$ is positive for every $\delta > 0$. Finally, for every $n \geq 1$, and $r \geq 1$,
\[
\left( \int_{B} \left\| \sum_{k=1}^{n} L_{n}(x) x_{L_{n}} P(dx) \right\|^{r}_{B} P(dx) \right)^{1/r} < \infty.
\]
From [1, Theorem 1] we also conclude that, as $n \to \infty$,
\[
\int_{B} \left\| x - \sum_{k=1}^{n} L_{k}(x) x_{L_{k}} \right\|^{r}_{B} P(dx) \to 0. \hfill \Box
\]

For applications of the Proposition see [3], [4]. In [6], Walsh applied the Chatterji Theorem [1, Theorem 1] in much the same way as here, to the Wiener measure case. The reproducing kernel representation (a) was not given however. More recently, Kuelbs [4] proved the existence of a representation of type (b) bypassing the Chatterji result, and hence avoiding the question of integrability of $\|x\|_{B}$. The role of reproducing kernels was not discussed. Finally, Jain and Kallianpur [3] gave still another proof bypassing the Chatterji result, showing the existence of certain embeddings of the reproducing kernel Hilbert space into $B$. The representation (a) was not given. Neither [3] nor [4] discuss the convergence in $r$th mean.

REFERENCES


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