A NOTE ON $\mathcal{I}$-REALCOMPACTIFICATIONS

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Abstract. Orrin Frink showed that the real-valued functions over a Tychonoff space $X$ which may be continuously extended to $\omega(\mathcal{I})$, the Wallman-type compactification associated with a normal base $\mathcal{I}$ for $X$, are those which are $\mathcal{I}$-uniformly continuous.

Let $\mathcal{I}$ be a delta normal base on a Tychonoff space $X$, and let $\eta(\mathcal{I})$ be the corresponding $\mathcal{I}$-realcompactification of $X$. In this note we show that countable $\mathcal{I}$-uniform continuity is a sufficient but not a necessary condition for continuously extending real-valued functions over $X$ to $\eta(\mathcal{I})$.

In [3], Orrin Frink utilized the notion of a normal base to obtain Hausdorff compactifications for Tychonoff or completely regular $T_1$ spaces $X$. A normal base $\mathcal{I}$ for the closed sets of a space $X$ is a base which is a disjunctive ring of sets, disjoint members of which may be separated by disjoint complements of members of $\mathcal{I}$. Frink proved that if $\mathcal{I}$ is a normal base for a $T_1$ space $X$, then the space $\omega(\mathcal{I})$ consisting of the $\mathcal{I}$-ultrafilters, is a Hausdorff compactification of $X$. By choosing different normal bases $\mathcal{I}$ for a noncompact space $X$, different Hausdorff compactifications of $X$ may be obtained.

In [1], Alo and Shapiro used $\mathcal{I}$-ultrafilters from a delta normal base (a normal base closed under countable intersections) to introduce a new space $\eta(\mathcal{I})$ consisting of those $\mathcal{I}$-ultrafilters with the countable intersection property. To each delta normal base $\mathcal{I}$ on $X$ there corresponds a delta normal base $\mathcal{I}^*$ on $\eta(\mathcal{I})$, and they have shown that every $\mathcal{I}^*$-ultrafilter with the countable intersection property is fixed, i.e., $\eta(\mathcal{I})$ is $\mathcal{I}^*$-realcompact. For many delta normal bases $\mathcal{I}$, $\eta(\mathcal{I})$ is realcompact in the usual sense but, in [5], it has been shown that this is not always the case.

A real function $f$ defined over a space $X$ with normal base $\mathcal{I}$ is said to be $\mathcal{I}$-uniformly continuous if for every positive epsilon there exists a finite open cover of $X$ by $\mathcal{I}$-complements, on each of which the oscillation of $f$ is less than epsilon. Frink showed that $f$ may be continuously extended to $\omega(\mathcal{I})$ if and only if $f$ is $\mathcal{I}$-uniformly continuous.

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615
The analogous condition for a space \( X \) with a delta normal base is

**countable \( \mathcal{L} \)-uniform continuity**: A real function \( f \) defined over a space \( X \) with delta normal base \( \mathcal{L} \) is countable \( \mathcal{L} \)-uniformly continuous if corresponding to every positive epsilon there exists a finite or denumerable open cover of \( X \) by \( \mathcal{L} \)-complements, on each of which the oscillation of \( f \) is less than epsilon. In this note we show that countable \( \mathcal{L} \)-uniform continuity is a sufficient but not a necessary condition for extendibility to \( \eta(\mathcal{L}) \).

For definitions and a thorough discussion of the results cited above, the reader is referred to [1] and [3].

**Theorem.** Every countable \( \mathcal{L} \)-uniformly continuous function on \( X \) can be continuously extended to a real-valued function on \( \eta(\mathcal{L}) \).

**Proof.** If \( \mathcal{L} \) is a delta normal base for \( X \), let \( \mathcal{U} \) be the collection of all free \( \mathcal{L} \)-ultrafilters on \( X \) with the countable intersection property. Then \( \eta(\mathcal{L})=X\cup\mathcal{U} \) and the topology for \( \eta(\mathcal{L}) \) is that obtained by taking as a base for the closed sets the family of all sets \( A^* \) of the form \( A\cup\{A\in\mathcal{U}|A\in\mathcal{A}\} \) where \( A\in\mathcal{L} \).

If \( f \) is a countable \( \mathcal{L} \)-uniformly continuous function on \( X \), we define a function \( g \) which extends \( f \) from \( X \) to \( \eta(\mathcal{L}) \) as follows. If \( x\in X \) we let \( g(x)=f(x) \). If \( \mathcal{A}\in\mathcal{U} \) then the family \( S_{\mathcal{A}}=\{f(A):A\in\mathcal{A}\} \) has the finite intersection property and is therefore a subbase for the filter \( \mathcal{F}_{\mathcal{A}} \) consisting of all supersets of finite intersections of members of \( S_{\mathcal{A}} \). The filter \( \mathcal{F}_{\mathcal{A}} \) is a Cauchy filter and therefore converges uniquely to a real number which we call \( g(\mathcal{A}) \).

That \( g \) is continuous at each point of \( X \) is readily verified. It remains to show that \( g \) is continuous at each point \( \mathcal{B}\in\mathcal{U} \). Let the family \( \{X-C_i\}_{i=1}^\infty \) be a denumerable cover of \( X \) by \( \mathcal{L} \)-complements, on each member of which the oscillation of \( f \) is less than \( \epsilon/3 \) (we lose no generality in assuming that the cover of \( X \) is denumerable). We may suppose that \( C_i\in\mathcal{B} \) so that there is an element \( Q\in\mathcal{B} \) with \( Q\subseteq X-C_i \). We show that

\[
g(\eta(\mathcal{L}) - C_i^*)
g((X - C_i) \cup \{\mathcal{A}\in\mathcal{U}:\exists P\in\mathcal{A} \text{ with } P \subseteq X - C_i\}) \subseteq S(g(\mathcal{B}), \epsilon).
\]

Now \( g(\mathcal{B})\in\text{cl}_Rf(Q) \) and we choose \( q\in Q \) so that \( |g(\mathcal{B})-f(q)|<\epsilon/3 \). If \( y\in X-C_i \) we then have

\[
|g(\mathcal{B}) - g(y)| \leq |g(\mathcal{B})-f(q)| + |f(q) - g(y)| < \epsilon/3 + \epsilon/3 < \epsilon.
\]

It therefore follows that \( g(X-C_i)\subseteq S(g(\mathcal{B}), \epsilon) \). If \( \mathcal{A}\in\mathcal{U} \) and there is a \( P\in\mathcal{A} \) with \( D\subseteq X-C_i \), we choose a point \( p\in P \) satisfying \( |g(\mathcal{A})-f(p)|<\epsilon/3 \).
The points \( q \) and \( p \) are members of \( X - C_1 \) and so

\[
|g(\mathcal{F}) - g(\mathcal{D})| \leq |g(\mathcal{F}) - f(p)| + |f(p) - f(q)| + |f(q) - g(\mathcal{D})| \\
< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.
\]

Thus \( g \) is a continuous, real-valued function on \( \eta(\mathcal{D}) \).

In case \( \mathcal{D} \) is the collection of all zero-sets of a Tychonoff space \( X \), then \( \eta(\mathcal{D}) \) is precisely the Hewitt realcompactification of \( X \). By observing that a real-valued function on a topological space is continuous if and only if it is countable zero-set uniformly continuous, we have the following well-known result [4] as a

**Corollary.** *Every continuous, real-valued function on a Tychonoff space \( X \) can be continuously extended to a real-valued function on \( X \), the Hewitt realcompactification of \( X \).*

However, countable \( \mathcal{D} \)-uniform continuity is not a necessary condition for extendibility. To see this, we will make use of an example given by A. Steiner and E. Steiner in [5].

Let \( X = [0, 1] \) with the discrete topology, let \( \mathcal{F}_1 \) be the family of all closed subsets of \( X \) with respect to the usual topology on \( [0, 1] \), and let \( \mathcal{F}_2 \) be the family of all subsets of \( X \) which are finite or whose complement is countable. Then the family \( \mathcal{D} \) of countable intersections of finite unions of members of \( \mathcal{F}_1 \cup \mathcal{F}_2 \) is a delta normal base. Furthermore, if \( \mathcal{A} \) is \( \mathcal{D} \)-ultrafilter, then \( \mathcal{A} \), being prime, contains a decreasing sequence of closed intervals whose lengths converge to 0; so if \( \mathcal{A} \) has the countable intersection property, then \( \mathcal{A} \) is fixed. Thus \( \eta(\mathcal{D}) = X \).

The \( \mathcal{D} \)-complements are all sets \( U \) of the form either:

(i) \( U \) is denumerable or finite;

(ii) \( U \) is open with respect to the usual topology on \([0, 1]\); or

(iii) \( U = V_1 \cup V_2 \) where \( V_1 \) has form (i) and \( V_2 \) form (ii). The function \( f: X \to \mathbb{R} \) equal to 1 on the rationals and 0 on the irrationals is certainly extendible to \( \eta(\mathcal{D}) = X \). However, if \( \{U_i\}_{i=1}^\infty \) is a cover of \( X \) by \( \mathcal{D} \)-complements, then at least one \( U_i \) has form (ii) or (iii). Hence \( f \) is not countable \( \mathcal{D} \)-uniformly continuous.

A delta normal base \( \mathcal{D} \) is a strong delta normal base if each \( A \in \mathcal{D} \) is a countable intersection of \( \mathcal{D} \)-complements. For such normal bases, \( \eta(\mathcal{D}) \) is always realcompact [2].

The normal base \( \mathcal{D} \) in the above example is easily seen to be strong delta. Hence, also in this situation, countable \( \mathcal{D} \)-uniform continuity is a sufficient but not a necessary condition for extendibility.
REFERENCES


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