A NOTE ON $\mathcal{D}$-REALCOMPACTIFICATIONS

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Abstract. Orrin Frink showed that the real-valued functions over a Tychonoff space $X$ which may be continuously extended to $\omega(\mathcal{D})$, the Wallman-type compactification associated with a normal base $\mathcal{D}$ for $X$, are those which are $\mathcal{D}$-uniformly continuous.

Let $\mathcal{D}$ be a delta normal base on a Tychonoff space $X$, and let $\eta(\mathcal{D})$ be the corresponding $\mathcal{D}$-realcompactification of $X$. In this note we show that countable $\mathcal{D}$-uniform continuity is a sufficient but not a necessary condition for continuously extending real-valued functions over $X$ to $\eta(\mathcal{D})$.

In [3], Orrin Frink utilized the notion of a normal base to obtain Hausdorff compactifications for Tychonoff or completely regular $T_1$ spaces $X$. A normal base $\mathcal{D}$ for the closed sets of a space $X$ is a base which is a disjunctive ring of sets, disjoint members of which may be separated by disjoint complements of members of $\mathcal{D}$. Frink proved that if $\mathcal{D}$ is a normal base for a $T_1$ space $X$, then the space $\omega(\mathcal{D})$ consisting of the $\mathcal{D}$-ultrafilters, is a Hausdorff compactification of $X$. By choosing different normal bases $\mathcal{D}$ for a noncompact space $X$, different Hausdorff compactifications of $X$ may be obtained.

In [1], Alo and Shapiro used $\mathcal{D}$-ultrafilters from a delta normal base (a normal base closed under countable intersections) to introduce a new space $\eta(\mathcal{D})$ consisting of those $\mathcal{D}$-ultrafilters with the countable intersection property. To each delta normal base $\mathcal{D}$ on $X$ there corresponds a delta normal base $\mathcal{D}^*$ on $\eta(\mathcal{D})$, and they have shown that every $\mathcal{D}^*$-ultrafilter with the countable intersection property is fixed, i.e., $\eta(\mathcal{D})$ is $\mathcal{D}^*$-realcompact. For many delta normal bases $\mathcal{D}$, $\eta(\mathcal{D})$ is realcompact in the usual sense but, in [5], it has been shown that this is not always the case.

A real function $f$ defined over a space $X$ with normal base $\mathcal{D}$ is said to be $\mathcal{D}$-uniformly continuous if for every positive epsilon there exists a finite open cover of $X$ by $\mathcal{D}$-complements, on each of which the oscillation of $f$ is less than epsilon. Frink showed that $f$ may be continuously extended to $\omega(\mathcal{D})$ if and only if $f$ is $\mathcal{D}$-uniformly continuous.

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The analogous condition for a space $X$ with a delta normal base is countable $\mathcal{L}$-uniform continuity: A real function $f$ defined over a space $X$ with delta normal base $\mathcal{L}$ is countable $\mathcal{L}$-uniformly continuous if corresponding to every positive epsilon there exists a finite or denumerable open cover of $X$ by $\mathcal{L}$-complements, on each of which the oscillation of $f$ is less than epsilon. In this note we show that countable $\mathcal{L}$-uniform continuity is a sufficient but not a necessary condition for extendibility to $\eta(\mathcal{L})$.

For definitions and a thorough discussion of the results cited above, the reader is referred to [1] and [3].

**Theorem.** Every countable $\mathcal{L}$-uniformly continuous function on $X$ can be continuously extended to a real-valued function on $\eta(\mathcal{L})$.

**Proof.** If $\mathcal{L}$ is a delta normal base for $X$, let $\mathcal{U}$ be the collection of all free $\mathcal{L}$-ultrafilters on $X$ with the countable intersection property. Then $\eta(\mathcal{L})=X\cup\mathcal{U}$ and the topology for $\eta(\mathcal{L})$ is that obtained by taking as a base for the closed sets the family of all sets $A^*$ of the form $A\cup\{A\in\mathcal{U}|A\in\mathcal{A}\}$ where $A\in\mathcal{L}$.

If $f$ is a countable $\mathcal{L}$-uniformly continuous function on $X$, we define a function $g$ which extends $f$ from $X$ to $\eta(\mathcal{L})$ as follows. If $x\in X$ we let $g(x)=f(x)$. If $s\in\mathcal{U}$ then the family $S_{sf}=\{f(A):A\in\mathcal{A}\}$ has the finite intersection property and is therefore a subbase for the filter $\mathcal{F}_{sf}$ consisting of all supersets of finite intersections of members of $S_{sf}$. The filter $\mathcal{F}_{sf}$ is a Cauchy filter and therefore converges uniquely to a real number which we call $g(s)$.

That $g$ is continuous at each point of $X$ is readily verified. It remains to show that $g$ is continuous at each point $s\in\mathcal{U}$. Let the family $\{X-C_i\}_{i=1}^{\infty}$ be a denumerable cover of $X$ by $\mathcal{L}$-complements, on each member of which the oscillation of $f$ is less than $\varepsilon/3$ (we lose no generality in assuming that the cover of $X$ is denumerable). We may suppose that $C_i\in\mathcal{B}$ so that there is an element $Q\in\mathcal{B}$ with $Q\subseteq X-C_i$. We show that

$$g[\eta(\mathcal{L}) - C_i] = g[(X-C_i) \cup \{A \in \mathcal{U}: \exists P \in \mathcal{A} \text{ with } P \subseteq X-C_i\}] \subseteq S(g(\mathcal{B}), \varepsilon).$$

Now $g(\mathcal{B}) \in cl_{R}[f(Q)]$ and we choose $q \in Q$ so that $|g(\mathcal{B}) - f(q)| < \varepsilon/3$. If $y \in X-C_i$ we then have

$$|g(\mathcal{B}) - g(y)| \leq |g(\mathcal{B}) - f(q)| + |f(q) - g(y)| < \varepsilon/3 + \varepsilon/3 < \varepsilon.$$

It therefore follows that $g(X-C_i) \subseteq S(g(\mathcal{B}), \varepsilon)$. If $A \in \mathcal{U}$ and there is a $P \in \mathcal{A}$ with $D \subseteq X-C_1$, we choose a point $p \in P$ satisfying $|g(A) - f(p)| < \varepsilon/3$. 

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The points $q$ and $p$ are members of $X - C_1$ and so

$$|g(\mathcal{A}) - g(\mathcal{B})| \leq |g(\mathcal{A}) - f(p)| + |f(p) - f(q)| + |f(q) - g(\mathcal{B})|$$

$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$ 

Thus $g$ is a continuous, real-valued function on $\eta(\mathcal{F})$.

In case $\mathcal{F}$ is the collection of all zero-sets of a Tychonoff space $X$, then $\eta(\mathcal{F})$ is precisely the Hewitt realcompactification of $X$. By observing that a real-valued function on a topological space is continuous if and only if it is countable zero-set uniformly continuous, we have the following well-known result [4] as a

**Corollary.** Every continuous, real-valued function on a Tychonoff space $X$ can be continuously extended to a real-valued function on $X$, the Hewitt realcompactification of $X$.

However, countable $\mathcal{F}$-uniform continuity is not a necessary condition for extendibility. To see this, we will make use of an example given by A. Steiner and E. Steiner in [5].

Let $X = [0, 1]$ with the discrete topology, let $\mathcal{F}_1$ be the family of all closed subsets of $X$ with respect to the usual topology on $[0, 1]$, and let $\mathcal{F}_2$ be the family of all subsets of $X$ which are finite or whose complement is countable. Then the family $\mathcal{F}$ of countable intersections of finite unions of members of $\mathcal{F}_1 \cup \mathcal{F}_2$ is a delta normal base. Furthermore, if $\mathcal{A}$ is $\mathcal{F}$-ultrafilter, then $\mathcal{A}$, being prime, contains a decreasing sequence of closed intervals whose lengths converge to 0; so if $\mathcal{A}$ has the countable intersection property, then $\mathcal{A}$ is fixed. Thus $\eta(\mathcal{F}) = X$.

The $\mathcal{F}$-complements are all sets $U$ of the form either:

(i) $U$ is denumerable or finite;

(ii) $U$ is open with respect to the usual topology on $[0, 1]$; or

(iii) $U = V_1 \cup V_2$ where $V_1$ has form (i) and $V_2$ form (ii). The function $f: X \rightarrow \mathbb{R}$ equal to 1 on the rationals and 0 on the irrationals is certainly extendible to $\eta(\mathcal{F}) = X$. However, if $\{U_i\}_{i=1}^\infty$ is a cover of $X$ by $\mathcal{F}$-complements, then at least one $U_i$ has form. (ii) or (iii). Hence $f$ is not countable $\mathcal{F}$-uniformly continuous.

A delta normal base $\mathcal{F}$ is a strong delta normal base if each $A \in \mathcal{F}$ is a countable intersection of $\mathcal{F}$-complements. For such normal bases, $\eta(\mathcal{F})$ is always realcompact [2].

The normal base $\mathcal{F}$ in the above example is easily seen to be strong delta. Hence, also in this situation, countable $\mathcal{F}$-uniform continuity is a sufficient but not a necessary condition for extendibility.
REFERENCES


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