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H-STRUCTURES ON PRODUCTS

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ABSTRACT. This note outlines a technique for determining the set of homotopy classes of H-structures on a product of H-spaces. The technique is then applied to computing the number of H-structures on $SO(4)$.

1. **Introduction.** This note outlines a technique for determining the set of homotopy classes of H-structures on a product $X=X_1\times X_2\times\cdots\times X_n$ of H-spaces. This technique is then applied to computing the number of H-structures on $SO(4)=SO(3)\times S^3$.

The technique may be summarized in the following statements.

(1) The set of homotopy classes of H-structures on X is in one-to-one correspondence with the homotopy set $[X\wedge X; X]$.

(2) There are subcomplexes $*=L_0\subset L_1\subset\cdots\subset L_{2n}=X\wedge X$ such that the sequences $0\to[L_r/L_{r-1}; X]\to[L_r; X]\to[L_{r-1}; X]\to 0$ are exact for $r=1, 2, \cdots, 2n$.

(3) $[L_r/L_{r-1}; X]$ is the direct product of the loops $[X_{i_1}\wedge\cdots\wedge X_{i_p}\wedge X_{j_1}\wedge\cdots\wedge X_{j_q}; X]$ with $p, q\geqq 1$ and $p+q=r$.

(4) For any pointed space L, $[L; X]$ is the direct product of the loops $[L; X_k]$ for $k=1, 2, \cdots, n$.

Statement (4) is completely routine and (1) is essentially Theorem 5.5A of **[1]**. Statements (2) and (3) are established in §2. Throughout this note all spaces should be assumed to be CW-complexes of finite type.

As a corollary to the statements (1)–(4) above, one sees that the number $\#[X\wedge X; X]$ of products on X is the product of the numbers

$$\#[X_{i_1}\wedge\cdots\wedge X_{i_p}\wedge X_{j_1}\wedge\cdots\wedge X_{j_q}; X_k].$$

2. **Main results.** We assume that each of the spaces X_k $(k=1, 2, \cdots, n)$ is a path-connected, pointed CW-complex of finite type and has at least one H-structure. As usual, $*$ will be used ambiguously to denote base-points. Let $X=X_1\times\cdots\times X_n$ and let $\phi: X\times X\to X\wedge X$ be the usual identification map. Suppose p, q are integers with $1\leqq p$, $q\leqq n$ and that $\alpha=(i_1,\cdots,i_p,j_1,\cdots,j_q)$ is a $(p+q)$-tuple of integers with $1\leqq i_1<i_2<\cdots<i_p\leqq n$

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and $1 \leqq j_1 < j_2 < \cdots < j_q \leqq n$. Such an α will be called *admissible*. Define $\#\alpha = p+q$ and let $K_\alpha \subset X \wedge X$ be the set of points $\phi(x_1, \cdots, x_n, x'_1, \cdots, x'_n)$ with $x_i = x_j = *$ except when $i = i_1, \cdots, i_p$ or $j = j_1, \cdots, j_q$. Let $L_r = \bigcup \{K_\alpha \mid \#\alpha = r\}$ for $r = 2, 3, \cdots, 2n$ and let $L_1 = * = \phi(X \vee X)$. Note that $L_1 \subset L_2 \subset \cdots \subset L_{2n} = X \wedge X$. Let $M_\alpha = X_{i_1} \wedge \cdots \wedge X_{i_p} \wedge X_{j_1} \wedge \cdots \wedge X_{j_q}$.

2.1 Lemma (James). *If $Y_1, \cdots, Y_m$ are subcomplexes of Y and if $r_i: Y \to Y_i$ are retractions with $r_i(Y_j) \subset Y_j$ for all i and j, then $Y_1 \cup \cdots \cup Y_m$ is retractile in Y* [**3**, Lemma 3.1].

2.2 Lemma. *If $r \leqq s$, then L_r is retractile in L_s.*

Proof. In view of [**8**, Proposition (3.7)] it suffices to show each L_r is retractile in $X \wedge X$. But the projections

$$X \times X \to X_{i_1} \times \cdots \times X_{i_p} \times X_{j_1} \times \cdots \times X_{j_q}$$

induce retractions $X \wedge X \to K_\alpha$ satisfying the hypothesis of James' Lemma.

2.3 Theorem. *The sequences*

$$0 \to [L_r/L_{r-1}; X] \to [L_r; X] \to [L_{r-1}; X] \to 0$$

are exact for $r = 2, 3, \cdots, 2n$.

This is an immediate consequence of O'Neill's work [**8**, Theorems (3.3) and (3.4)]. Note that if X is the space of loops on an H-space, then a result of Eckmann and Hilton [**2**, Theorem 2.2] implies that these sequences split.

2.4 Theorem. *$[L_r/L_{r-1}; X]$ is the direct product of the loops $[M_\alpha; X]$ with $\#\alpha = r$.*

Proof. It suffices to establish that $L_r/L_{r-1} = \vee\{M_\alpha \mid \#\alpha = r\}$. But if $\alpha \neq \beta$ are admissible r-triples, then $K_\alpha \cap K_\beta \subset L_{r-1}$. The result now follows from the observation that $K_\alpha/(K_\alpha \cap L_{r-1}) = M_\alpha$.

3. Application to $SO(4)$.

3.1 Theorem. *The space $X = SO(4)$ has $2^{86} \cdot 3^{16}$ homotopy classes of H-structures.*

In the course of the proof it will be convenient to refer to the following table of values of $[\Sigma^k P^2; S^3]$.

Table 3.2

$k =$	2	3	4	5	6	7	8	9
$[\Sigma^k P^2; S^3] =$	(2)	(4)	(2) + (2)	(2) + (2)	(4)	(2)	(1)	(2)

The symbol (m) denotes the abelian group of integers modulo m, and $+$ denotes direct sums. The entries are readily deduced from the Appendix to Chapter 12 of Hilton's book [4].

PROOF OF 3.1. We have $X_1=P^3=SO(3)$, $X_2=S^3$. Note that since $X\wedge X$ is simply-connected, $[X\wedge X; S^3]$ is isomorphic to $[X\wedge X; X_1]$ under the 2-fold covering projection $S^3\to P^3$. Let

$$\begin{aligned} a &= \#[X_1 \wedge X_2 \wedge X_1 \wedge X_2; S^3] = \#[S^6 \wedge P^3 \wedge P^3; S^3],\\ b_1 &= \#[X_1 \wedge X_1 \wedge X_2; S^3] = \#[S^3 \wedge P^3 \wedge P^3; S^3],\\ b_2 &= \#[X_1 \wedge X_2 \wedge X_2; S^3] = \#[S^6 \wedge P^3; S^3];\\ c_1 &= \#[X_1 \wedge X_1; S^3],\\ c_2 &= \#[X_1 \wedge X_2; S^3] = \#[S^3 \wedge P^3; S^3] \quad \text{and}\\ c_3 &= \#[X_2 \wedge X_2; S^3] = \#[S^6; S^3]; \end{aligned}$$

then $\#[X\wedge X; X]=a^2b_1^4b_2^4c_1^2c_2^4c_3^2$. The evaluation of these numbers makes repeated use of Naylor's theorem that $S^3\wedge P^3=(S^3\wedge P^2)\vee S^6$ [6]. In addition, we will use Table 3.2 several times in each calculation.

The value of a.

$$S^6 \wedge P^3 \wedge P^3 = ((S^3 \wedge P^2) \vee S^6) \wedge ((S^3 \wedge P^2) \vee S^6).$$

Thus

$$a = \#[S^6 \wedge P^2 \wedge P^2; S^3] \cdot (\#[S^9 \wedge P^2; S^3])^2 \cdot \#[S^{12}; S^3].$$

The cofibration sequence of spaces

$$S^1 \xrightarrow[f]{} S^1 \longrightarrow P^2 \longrightarrow S^2 \xrightarrow[f]{} S^2$$

in which f ambiguously denotes maps of degree 2, induces the cofibration sequence

$$\Sigma^7 P^2 \xrightarrow[f]{} \Sigma^7 P^2 \longrightarrow S^6 \wedge P^2 \wedge P^2 \longrightarrow \Sigma^8 P^2 \xrightarrow[f]{} \Sigma^8 P^2$$

and hence the exact sequence of groups

$$\xleftarrow[f^*]{} [\Sigma^7 P^2; S^3] \longleftarrow [S^6 \wedge P^2 \wedge P^2; S^3] \longleftarrow [\Sigma^8 P^2; S^3] \xleftarrow[f^*]{}.$$

The two outer groups are repetitions and have been omitted from the line above. Since $[\Sigma^7P^2; S^3]=Z_2$ and $[\Sigma^8P^2; S^3]=0$, both homomorphisms f^* are trivial, and $\#[S^6\wedge P^2\wedge P^2; S^3]=2$.

On the other hand, $\#[\Sigma^9P^2; S^3]=2$ and $\#[S^{12}; S^3]=4$. Thus $a=2^5$.

The value of b_1.

$$\begin{aligned} S^3 \wedge P^3 \wedge P^3 &= ((S^3 \wedge P^2) \wedge S^6) \wedge P^3 = (P^2 \wedge S^3 \wedge P^3) \vee S^6 \wedge P^3\\ &= (S^3 \wedge P^2 \wedge P^2) \vee (\Sigma^6 P^2) \vee (\Sigma^6 P^2) \vee S^9. \end{aligned}$$

Thus $b_1=\#[S^3\wedge P^2\wedge P^2;\ S^3]\cdot(\#[\Sigma^6P^2;\ S^3])^2\cdot\#[S^9;\ S^3]$. As before, we have an exact sequence

$$\underset{f^*}{\longleftarrow}[\Sigma^4P^2;\ S^3]\longleftarrow[S^3\wedge P^2\wedge P^2;\ S^3]\longleftarrow[\Sigma^5P^2;\ S^3]\underset{f^*}{\longleftarrow}.$$

But $[\Sigma^4P^2;\ S^3]\approx[\Sigma^5P^2;\ S^3]\approx Z_2+Z_2$, and f^* is trivial on each of these. This shows $\#[S^3\wedge P^2\wedge P^2;\ S^3]=2^4$.

On the other hand $\#[\Sigma^6P^2;\ S^3]=2^2$ and $\#\pi_9(S^3)=3$, whence $b_1=2^8\cdot 3$.

The values of b_2, c_1, c_2 *and* c_3.

$$b_2=\#[S^6\wedge P^3;S^3]=\#[S^6\wedge P^2;S^3]\cdot\#[S^9;S^3]=2^2\cdot 3;$$

$$c_1=\#[P^3\wedge P^3;S^3]=2^8\cdot 3\quad [\mathbf{4}];$$

$$c_2=\#[S^3\wedge P^3;S^3]=\#[S^3\wedge P^2;S^3]\cdot\#[S^6;S^3]=4\cdot 12=2^4\cdot 3;$$

$$c_3=\#\pi_6(S^3)=2^2\cdot 3.$$

Thus $\#[X\wedge X;\ X]=2^{86}\cdot 3^{16}$.

A similar calculation would show that $S^3\times S^3$ has $2^{20}\cdot 3^{16}$ homotopy classes of H-structures. This result was obtained earlier by Norman [**7**] using Loibel's Theorem [**5**].

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