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HOMOTOPICAL NILPOTENCE OF THE SEVEN SPHERE

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ABSTRACT. We prove that the homotopical nilpotence of S^7 is 3, with respect to any of its 120 H-space multiplications.

The homotopical nilpotence of S^3 has been calculated by Porter [**4**] for the standard multiplication and by Arkowitz and Curjel [**1**] for all of its twelve H-space multiplications. Arkowitz and Curjel mention that their methods lead to results on the multiplications on S^7 but do not calculate its homotopical nilpotence. By modifying their arguments with the Samelson products we obtain the results on S^7 easily.

We will denote the collection of homotopy classes of basepoint preserving maps from A to B by $[A, B]$ and we will not distinguish notationally between a map and its homotopy class. The multiplication and inverse in the unit Cayley numbers induce the standard multiplication $m\in[S^7\times S^7, S^7]$ and two sided homotopy inverse $\nu\in[S^7, S^7]$ on the space S^7. For the H-space (S^7, m, ν) we define a *commutator map* $\phi: S^7\times S^7\to S^7$ by $\phi(x,y)=(x\cdot y)\cdot(x^{-1}\cdot y^{-1})$ using the multiplication m and inverse ν. Recall that the Cayley multiplication is not associative but is diassociative, i.e. any two elements generate an associative subalgebra. We now make a choice in bracketing to define inductively the *k-fold commutator map* $\phi:(S^7)^k\to S^7$ by $\phi_k=\phi\circ(\phi_{k-1}\times 1)$ where $\phi_1=1$, the identity map on S^7. It is well known that ϕ_k induces a unique homotopy class $\psi_k\in[\bigwedge^k S^7, S^7]$ with $\psi_k\circ q_k=\phi_k$, where $\bigwedge^k S^7$ is the k-fold smash product of S^7 (homeomorphic to S^{7k}) and $q_k:(S^7)^k\to\bigwedge^k S^7$ is the projection map. The *homotopical nilpotence* of the H-space (S^7, m, ν) written $\mathrm{nil}(S^7, m, \nu)$, is the least integer k such that ϕ_{k+1} (and hence ψ_{k+1}) is nullhomotopic.

THEOREM. $\mathrm{nil}(S^7, m, \nu)=3$.

PROOF. James [**2**, p.176] proves that ψ_2 generates $\pi_{14}(S^7)=\boldsymbol{Z}_{120}$ so that in Toda's notation [**5**] its 2-component is σ', its 3-component is $\alpha_2(7)$ and its 5-component is $\alpha_1(7)$. Now

$$\psi_3=\psi\circ(\psi\wedge 1)=\psi\circ\Sigma^7\psi\in\pi_{21}(S^7)=\boldsymbol{Z}_{24}\oplus\boldsymbol{Z}_4$$

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and its 2-component is $\sigma' \circ \Sigma^7 \sigma' = 2\sigma' \circ \sigma_{14} \neq 0$ [**5**]. The element α_2 is defined in terms of a Toda bracket and so the 3-component of ψ_3 is

$$\alpha_2(7) \circ \alpha_2(14) \in \{\alpha_1(7), 3\iota_{10}, \alpha_1(10)\} \circ \alpha_2(14) \subset \{\alpha_1(7), 3\iota_{10}, \alpha_1(10) \circ \alpha_2(13)\} = 0$$

since $\alpha_1(10) \circ \alpha_2(13) = 0$ by Lemma 13.8 of [**5**]. Hence ψ_3 has only a 2-component and

$$\psi_4 = \psi_3 \circ \Sigma^{14}\psi \in \pi_{28}(S^7) = \boldsymbol{Z}_6 \oplus \boldsymbol{Z}_2 \quad \text{by [3]}$$

and so $\psi_4 = 4\sigma' \circ \sigma_{14} \circ \sigma_{21} = 0$ which proves the theorem.

There are 120 different homotopy classes of multiplications on S^7 and as in Lemma 2 of [**1**] it can be shown that they can be written additively in the form

$$m^{(t)} = m + t\phi \in [S^7 \times S^7, S^7], \qquad t = 0, 1, \cdots, 119.$$

Also as in Lemma 3 of [**1**], ν is a homotopy inverse for each of these multiplications.

COROLLARY. $\operatorname{nil}(S^7, m^{(t)}, \nu) = 3$ *for* $t = 0, 1, \cdots, 119$.

PROOF. Denote by $\psi_k^{(t)} \in [\bigwedge^k S^7, S^7]$ the r-fold smash commutator map defined on the H-space $(S^7, m^{(t)}, \nu)$. Then James [**2**, p. 176] and Arkowitz and Curjel [**1**, Lemma 4] prove that $\psi_k^{(t)} = (2t+1)\psi_k$. Hence $\psi_3^{(t)}$ is nonzero and $\psi_4^{(t)}$ is zero, which proves the corollary.

Changing the choice of bracketing in the definition of the k-fold commutator map will at most affect a sign change in ψ_k, so that the homotopy nilpotence is independent of the choice of bracketing.

REFERENCES

1. M. Arkowitz and C. R. Curjel, *Some properties of the exotic multiplications on the three-sphere*, Quart. J. Math. Oxford Ser. (2) **20** (1969), 171–176. MR **40** #2066.
2. I. M. James, *On H-spaces and their homotopy groups*, Quart. J. Math. Oxford Ser. (2) **11** (1960), 161–179. MR **24** #A2966.
3. M. Mimura, *On the generalized Hopf homomorphism and the higher composition*. II, $\pi_{n+i}(S^n)$ *for* $i=21$ *and* 22, J. Math. Kyoto Univ. **4** (1965), 301–326. MR **31** #1676.
4. G. J. Porter, *Homotopical nilpotence of* S^3, Proc. Amer. Math. Soc. **15** (1964), 681–682. MR **29** #6496.
5. H. Toda, *Composition methods in homotopy groups of spheres*, Ann. of Math. Studies, no. 49, Princeton Univ. Press, Princeton, N.J., 1962. MR **26** #777.

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