POSITIVE TRANSFORMATIONS RESTRICTED TO SUBSPACES AND INEQUALITIES AMONG THEIR PROPER VALUES

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Abstract. Let $A$ be a positive Hermitian transformation on an $n$-dimensional unitary space $E_n$ with proper values $a_1 \geq \cdots \geq a_n$. Let $b_1 \geq \cdots \geq b_n$ be the proper values of $A|M$, where $M$ is a proper subspace of $E_n$ and $c_1 \geq \cdots \geq c_k$ be the proper values of $A|M^\perp$. Let $i_1 < \cdots < i_r$ and $j_1 < \cdots < j_r$ be sequences of positive integers, with $i_r \leq k$ and $j_r \leq h$. Then $(b_{i_1} \cdots b_{i_r})(c_{j_1} \cdots c_{j_r}) \geq (a_{i_1+j_1-1} \cdots a_{i_r+j_r-1})$. In this article generalizations of this inequality have been studied.

Let $A$ be a positive Hermitian linear transformation on a unitary space $E_n$ with proper values $a_1 \geq \cdots \geq a_n$. Let $M$ be a proper subspace of $E_n$. Let the proper values of $A|M$ be $b_1 \geq \cdots \geq b_n$ and the proper values of $A|M^\perp$ be $c_1 \geq \cdots \geq c_k$. Then N. Aronszajn [4] has given the inequality $a_1+\cdots+c_j \geq b_1+\cdots+b_j$ for $1 \leq i \leq h$ and $1 \leq j \leq k$. Generalizations of this inequality have been given by A. J. Hoffman, R. C. Thompson and L. J. Freede [5]. All of these inequalities involve sums of proper values. In this article we shall give generalizations of these inequalities containing products of proper values.

1. Definitions and notations. The inner product of two vectors $\alpha$ and $\beta$ will be denoted by $(\alpha, \beta)$. The determinant of a linear transformation $A$ on $E_n$ will be denoted by $\det A$. The identity transformation will be denoted by $I$. A Hermitian linear transformation is called positive if $(A\xi, \xi) > 0$ for all $\xi \neq 0$. An orthonormal set $\{x_1, \cdots, x_k\}$ will be indicated by $\{x_p\}$ o.n. The subspace spanned by the set $\{x_1, \cdots, x_k\}$ will be denoted by $[x_1, \cdots, x_k]$. We write $\dim M = h$ if the dimension of the subspace $M$ is $h$.

If $A$ is a linear transformation on a unitary space $E_n$ and if $M$ is a subspace of $E_n$, then we define a linear transformation $A|M$ as follows: if $\xi \in M$, let $[A|M]\xi = PA\xi$, where $P$ is the orthogonal projection on $M$. We observe that if $\alpha$ and $\beta \in M$, then $([A|M]\alpha, \beta) = (PA\alpha, \beta) = (A\alpha, \beta)$. It follows that if $A$ is Hermitian (positive), then so is $A|M$.

If $j_p \leq i_p$, for $p = 1, \cdots, k$, we write $(j_1, \cdots, j_k) \leq (i_1, \cdots, i_k)$ and say the sequence $(i_1, \cdots, i_k)$ is greater than or equal to the sequence $(j_1, \cdots, j_k)$.
(j_1, \cdots, j_k). Further, given any sequence i_1 \leq \cdots \leq i_k of positive integers, we define (i_1', \cdots, i_k') to be the strictly increasing sequence of positive integers such that (i_1', \cdots, i_k') \leq (i_1, \cdots, i_k) and (i_1', \cdots, i_k') \leq (j_1, \cdots, j_k), if (j_1, \cdots, j_k) is a strictly increasing sequence of positive integers greater than or equal to (i_1, \cdots, i_k) [1].

2. Some theorems. Let A be a positive transformation on \( E_n \) with proper values \( \lambda_1 \leq \cdots \leq \lambda_n \). Then

\[
(1) \quad m_1 \cdots m_k = \sup_{\xi_1 \in \mathbb{R}} \det((A_{\xi_1}, \xi_j)).
\]

This theorem is due to Ky Fan [3]. Further, if \( i_1 \leq \cdots \leq i_k \) is a sequence of positive integers such that \( i_p \leq n - k + p \), for \( p = 1, \cdots, k \), and \( k \leq n \), then

\[
(2) \quad \inf_{M_1 \subset \cdots \subset M_k} \sup_{\xi_1 \in \mathbb{R}} \det((A_{\xi_1}, \xi_j))_{1 \leq i_1 \leq \cdots \leq i_k} = m_{i_1} \cdots m_{i_k},
\]

where \( a_p \) stands for \( \dim M_p = i_p - 1 \) and \( M_p \) is a subspace of \( E_n \) [1].

If \( A \) is a Hermitian linear transformation on \( E_n \) with proper values \( \lambda_1 \leq \cdots \leq \lambda_n \) and \( t_1 \leq \cdots \leq t_k \) are the proper values of \( A| M \), where \( M \) is a subspace of \( E_n \) and \( \dim M = k \), then

\[
(3) \quad p_{n-k+i} \leq t_i \leq p_i,
\]

for \( i = 1, \cdots, k \) [2].

3. Theorem. Let \( A \) be a positive transformation on \( E_n \) with proper values \( \lambda_1 \leq \cdots \leq \lambda_n \). Let \( R_1, \cdots, R_s \) be proper subspaces of \( E_n \) such that \( R_i \) is orthogonal to \( R_j \), for \( i \neq j \), \( E_n = R_1 \oplus \cdots \oplus R_s \), and \( \dim R_q = h_q \), for \( q = 1, \cdots, s \). Suppose the proper values of \( A| R_q \) are \( \eta_1 \leq \cdots \leq \eta_{h_q} \), \( q = 1, \cdots, s \). Let \( i_1 \leq \cdots \leq i_{r+q} \), \( q = 1, \cdots, s \), be sequences of positive integers such that \( i_{qs} \leq h_q - r + p \), for \( q = 1, \cdots, r, p = 1, \cdots, r \), \( q = 1, \cdots, s \), with \( r \) less than or equal to the \( \min(h_1, \cdots, h_s) \). Then

\[
(1) \quad \prod_{q=1}^{s} \left( \prod_{p=1}^{r} b_{q,i_{qs}} \right) \geq \left( \prod_{p=1}^{n-r(s-1)+1} a_p \right) \left( \prod_{p=1}^{r} a_{v_p} \right),
\]

where \( v_p = (1-s+\sum_{q=1}^{s} i_{qs})^r \).

**Proof.** By \( \S 2 \) (2) there exist subspaces \( M_{q_1} \subset \cdots \subset M_{q_s} \subset R_q \), \( q = 1, \cdots, s \), with \( \dim M_{q_p} = i_q - 1 \), \( p = 1, \cdots, r \), \( q = 1, \cdots, s \), such that

\[
(2) \quad \sup_{M_{q_p} \perp M_{q_s} \subset R_q} \det((A| M_{q_p})_{\eta_{q_1}, \eta_{q_s}})_{1 \leq i_1 \leq \cdots \leq i_r} \text{ for } q = 1, \cdots, s.
\]
Let $L_p = M_1 \oplus \cdots \oplus M_p$, $p = 1, \ldots, r$. We observe that $L_1 \subset \cdots \subset L_r \subset E_n$ and $\dim L_p = (1 - s + \sum_{p=1}^{p-1} 1_{kp}) - 1$, $p = 1, \ldots, r$. Let $\{\zeta_1, \ldots, \zeta_r\}$ be an orthonormal set in $E_n$ such that $\zeta_p \perp L_p, p = 1, \ldots, r$. Now, for each $p = 1, \ldots, r$, there exists an orthonormal set $\{\eta_{1p}, \ldots, \eta_{sp}\}$ such that $\zeta_p \in [\eta_{1p}, \ldots, \eta_{sp}]$ and $\eta_{qp} \in M_q \cap R_q, q = 1, \ldots, s$. It is clear that there exists an orthonormal set $\{\eta_{11}, \ldots, \eta_{1r}, \eta_{21}, \ldots, \eta_{2s}, \ldots, \eta_{sr}\}$ such that $\eta_{qp} \in M_q \cap R_q, q = 1, \ldots, s, p = 1, \ldots, r$, with

$$[\eta_{11}, \ldots, \eta_{1r}, \ldots, \eta_{s1}, \ldots, \eta_{sr}] \subset [\eta_{11}, \ldots, \eta_{1r}, \ldots, \eta_{s1}, \ldots, \eta_{sr}].$$

We extend $\{\zeta_1, \ldots, \zeta_r\}$ to an orthonormal set $\{\xi_1, \ldots, \xi_r\}$ in such a way that $L = [\xi_1, \ldots, \xi_{1r}, \ldots, \xi_{s1}, \ldots, \xi_{sr}]$. Thus

$$\begin{pmatrix}
(A\eta_{11}, \eta_{11}) & \cdots & (A\eta_{11}, \eta_{sr}) \\
\vdots & & \vdots \\
(A\eta_{sr}, \eta_{11}) & \cdots & (A\eta_{sr}, \eta_{sr})
\end{pmatrix}$$

$$= \det(A | L) = \det((A\xi_1, \xi_j))_{1 \leq i \leq s, 1 \leq j \leq r}.$$ 

Consequently

$$\prod_{q=1}^{s} \det((A\eta_{q1}, \eta_{q1}))_{1 \leq i \leq s, 1 \leq j \leq r} \geq \det((A\xi_1, \xi_j))_{1 \leq i \leq s, 1 \leq j \leq r}.$$ 

Suppose $d_1 \geq \cdots \geq d_s$ are the proper values of $A|L$. By §2 (1) we obtain

$$d_1 \cdots d_r \geq ((A | L) \xi_1, \xi_1))_{1 \leq i \leq s, 1 \leq j \leq r} = \det((A\xi_1, \xi_j))_{1 \leq i \leq s, 1 \leq j \leq r}.$$ 

By §2 (3), it follows that

$$d_{r+1} \cdots d_s \geq \prod_{p=n-r(s-1)+1}^{n} \xi_p.$$ 

Combining (4), (5) and (6) we obtain

$$\prod_{q=1}^{s} \det((A\eta_{q1}, \eta_{q1}))_{1 \leq i \leq s, 1 \leq j \leq r}$$

$$\geq \left( \prod_{p=n-r(s-1)+1}^{n} \xi_p \right) \det((A\xi_1, \xi_j))_{1 \leq i \leq s, 1 \leq j \leq r}.$$ 

Using (2) and (7) we obtain

$$\prod_{q=1}^{s} \left( \prod_{p=1}^{\tilde{r}} b_{q, \tilde{i}_{qs}} \right)$$

$$\geq \left\{ \prod_{p=n-r(s-1)+1}^{n} \min \left\{ \inf \left( \sup_{K_1 \subset \cdots \subset K_r} \min_{L_1 \oplus \cdots \oplus L_r, (L_p)_{0,n}} \det((A\delta_1, \delta_j))_{1 \leq i \leq s, 1 \leq j \leq r} \right) \right\}.$$
where \( w_p \) stands for \( \dim K_p = (1 - s + \sum_{q=1}^{s} i_{qp}) - 1 \). But by §2 (2) we obtain

\[
\inf_{K_1 \subset \cdots \subset K_r} \sup_{w_p} \det((A_{ij}, \delta_{i} j))_{1 \leq i \leq s, 1 \leq j \leq r} = \prod_{p=1}^{r} a_{vp}
\]

where \( w_p \) stands for \( \dim K_p = (1 - s + \sum_{q=1}^{s} i_{qp}) - 1 \) and

\[
v_p = \left(1 - s + \sum_{q=1}^{s} i_{qp}\right)^{m}.
\]

Combining (8) and (9) we obtain (1); thus the proof is complete.

Indeed this theorem is true for a nonnegative transformation on \( E_n \).

4. COROLLARY. Let \( H \) be a Hermitian transformation on \( E_n \) with proper values \( a_1 \geq \cdots \geq a_n \). Let \( a \) be any real number such that \( a \leq a_n \). Then it is clear that \( H - aI \) is nonnegative. Let us consider subspaces and sequences of positive integers of §3. Let the proper values of \( H |_{R_q} \) be \( b_{q1} \geq \cdots \geq b_{qn} \), \( q = 1, \cdots, s \). Then applying §3 to \( H - aI \) we obtain

\[
\left( \prod_{q=1}^{s} \left( \prod_{p=1}^{r} (b_{q,p} - a) \right) \right) \geq \left( \prod_{p=n-r(s-1)+1}^{n} (a_p - a) \right) \prod_{p=1}^{r} [a_{vp} - a]
\]

where \( v_p = (1 - s + \sum_{q=1}^{s} i_{qp})^{m} \).

5. DEFINITION. Let \( i_1 \leq \cdots \leq i_k \) be a sequence of positive integers such that \( i_p \geq p \), for \( p = 1, \cdots, k \). We define \( (i_1', \cdots, i_k') \) to be the strictly increasing sequence of positive integers such that \( (i_1', \cdots, i_k') \leq (i_1, \cdots, i_k) \) and \( (j_1, \cdots, j_k) \leq (i_1', \cdots, i_k') \), if \( (j_1, \cdots, j_k) \) is a strictly increasing sequence of positive integers which is less than or equal to \( (i_1, \cdots, i_k) \).

6. REMARK. For a Hermitian linear transformation R. C. Thompson and L. J. Freede [5] have shown that

\[
\sum_{q=1}^{s} \left( \sum_{p=1}^{r} b_{q,p} \right) \leq \left( \sum_{p=1}^{r} a_p \right) + \sum_{p=1}^{s} a_{vp},
\]

where \( x_p = (\sum_{q=1}^{s} i_{qp})^{m} \) and where the symbols are defined in §3 except the obvious changes are made in the conditions on the sequences \( i_{q1}, \cdots, i_{qr} \). Thus we might expect a similar inequality for products. But this conjecture is refuted by the following example. Let \( A \) be represented by

\[
\begin{pmatrix}
9 & 1 \\
1 & 1 \\
\end{pmatrix}
\]

It is clear that \( A \) is positive. Consider the subdivision

\[
\begin{pmatrix}
9 & 1 \\
1 & 1 \\
\end{pmatrix}
\]

Then \( b_{1,1} \cdot b_{2,1} \cdot b_{1,1} = 9 > 8 = a_1 a_2 \).
REFERENCES


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