PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 32, Number 2, April 1972

ON THE GROUP INDICES OF THE PARASYMPLECTIC GROUP OF LEVEL F

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ABSTRACT. We obtain the group indices of a parasymplectic group to a principal congruence subgroup of level $\mathfrak{q}$ and to an inhomogeneous congruence subgroup of level $\mathfrak{q}$.

1. **Introduction.** Let $\mathfrak{o}$ be the integer ring of an algebraic number field. Let $\{f_i\}$ be a set of n elements in $\mathfrak{o}$ such that f_i is divisible by f_{i-1} for every i $(2\leqq i\leqq n)$. Then we define $f_{ij}=f_i^{-1}f_j$ for $i\leqq j$ and $f_{ij}=1$ for $i>j$, and denote the set $\{f_{ij}\}$ by F. When all $f_{ij}=1$, then F is denoted by I.

Let $\mathfrak{o}_n^*(F)$ be the ring of matrices $(f_{ij}x_{ij})$ of degree n, whose (i,j) element $f_{ij}x_{ij}$ belongs to the ideal $f_{ij}\mathfrak{o}$ of $\mathfrak{o}$ for every i,j. Then we denote $(f_{ij}x_{ji})$, whose (i,j) element is $f_{ij}x_{ji}$, by $\hat{X}$ for any matrix $X=(f_{ij}x_{ij})$ in $\mathfrak{o}_n^*(F)$.

Let $\mathfrak{o}_n(F)$ be the ring of matrices $\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)$ of degree $2n$, whose submatrices A, B, C, D belong to $\mathfrak{o}_n^*(F)$. We define a matrix $\hat{M}$ in $\mathfrak{o}_n(F)$ by

$$\begin{pmatrix} \hat{A} & \hat{C} \\ \hat{B} & \hat{D} \end{pmatrix}$$

associated with $M=\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)$ in $\mathfrak{o}_n(F)$.

Now we shall define three groups in $\mathfrak{o}_n(F)$ (see [**1**]); $\Gamma(n,F)=\{M \mid \hat{M}JM=J,\ M\in\mathfrak{o}_n(F)\}$, where

$$J=\begin{pmatrix} & E_n \\ -E_n & \end{pmatrix}$$

and E_n is the unit matrix in $\mathfrak{o}_n^*(F)$ (the parasymplectic group of level F), $\Gamma(n, F, \mathfrak{q})=\{M \mid M\in\Gamma(n, F)$ and $M\equiv E_{2n} \bmod \mathfrak{q}\}$ where $\mathfrak{q}$ is an ideal in $\mathfrak{o}$ (the principal congruence subgroup of level $\mathfrak{q}$ of $\Gamma(n, F)$), $\Gamma^0(n, F, \mathfrak{q})=\{M \mid M\in\Gamma(n, F)$ and $C\equiv 0 \bmod \mathfrak{q}\}$, where $M=\left(\begin{smallmatrix} * & * \\ C & *\end{smallmatrix}\right)$ (the inhomogeneous congruence subgroup of level $\mathfrak{q}$ of $\Gamma(n, F)$). Then $\Gamma(n, F, \mathfrak{q})$ is a normal subgroup of $\Gamma(n, F)$.

Received by the editors January 5, 1971.

AMS 1969 *subject classifications*. Primary 2065, 1545.

Key words and phrases. Parasymplectic group, principal congruence subgroup of level $\mathfrak{q}$, inhomogeneous congruence subgroup of level $\mathfrak{q}$.

[1] This work was done while the author was at the University of Toronto, Toronto, Canada. The author acknowledges support received from the National Research Council of Canada, Grant No. A-7210.

Our principal aims of the paper are to prove the following theorems:

THEOREM 1. *The group* $\Gamma(n, F)/\Gamma(n, F, \mathfrak{q})$ *is isomorphic to a group* $\Delta(n, F, \mathfrak{q})$.

A definition of the group $\Delta(n, F, \mathfrak{q})$ is in §2 and a proof of the theorem is in §3 which is done by using a suitably modified argument of Klingen [3] and a "Hilfssatz" of Christian [1].

THEOREM 2. *For the group indices* $\mu(n, F, \mathfrak{q})=(\Gamma(n, F):\Gamma(n, F, \mathfrak{q}))$ *and* $\mu^0(n, F, \mathfrak{q})=(\Gamma(n, F):\Gamma^0(n, F, \mathfrak{q}))$ *it holds that*

$$\mu(n, F, \mathfrak{q}) = N(\mathfrak{q})^{n(2n+1)} \prod_{\mathfrak{p}|\mathfrak{q}} \prod_{i=1}^{n} (1 - N(\mathfrak{p}^{-2k_{i,\mathfrak{p}}})),$$

$$\mu^0(n, F, \mathfrak{q}) = N(\mathfrak{q})^{n(n+1)/2} \prod_{\mathfrak{p}|\mathfrak{q}} \prod_{i=1}^{n} (1 + N(\mathfrak{p}^{-k_{i,\mathfrak{p}}})),$$

where $N(\cdot)$ *is the norm of ideal, and* $k_{i,\mathfrak{p}}=k-i+1$ *is the largest integer for a fixed integer i and a prime ideal* $\mathfrak{p}$ *with* $\langle f_{ik}, \mathfrak{p}\rangle=1$.

From these theorems we shall get

COROLLARY OF THEOREM 2. *The group* $\Delta(n, F, \mathfrak{q})$ *is isomorphic to the group* $\Delta(n, I, \mathfrak{q})$, *when* $\langle f_{1i}, \mathfrak{q}\rangle=1$ *for every i.*

Proofs of Theorem 2 and the Corollary are in §4.

As our special cases we obtain $\mu(n, I, \mathfrak{q})$ and $\mu^0(n, I, \mathfrak{q})$ of Klingen [3], and those in which $\mathfrak{o}$ is the rational integer ring (Koecher [4] or Satake [5]).

The author wishes to thank Professor G. de B. Robinson for his kind support of this work.

2. **The group** $\Delta(n, F, \mathfrak{q})$. Let us denote the residue ring $f_{ij}\mathfrak{o}/f_{ij}\mathfrak{q}$ by $\mathfrak{o}_{ij}$ and let $\mathfrak{o}_n^*(F, \mathfrak{q})$ be the set of matrices of degree n, $(\bar{m}_{ij})$, with the element $\bar{m}_{ij}$ in $\mathfrak{o}_{ij}$.

We define an addition $\oplus$ and a multiplication $\otimes$ for any two elements $(\bar{m}_{ij})$ and $(\bar{n}_{ij})$ in $\mathfrak{o}_n^*(F, \mathfrak{q})$ by

$$(\bar{m}_{ij}) \oplus (\bar{n}_{ij}) = ((m_{ij} + n_{ij})^{-}), \qquad (\bar{m}_{ij}) \otimes (\bar{n}_{ij}) = \left(\sum_{j=1}^{n} \bar{m}_{ij} \cdot \bar{n}_{jk}\right),$$

where $\bar{m}_{ij}\cdot\bar{n}_{jk}=m_{ij}n_{jk} \bmod f_{ik}\mathfrak{q}$ for $m_{ij}\in f_{ij}\mathfrak{o}$, $n_{jk}\in f_{jk}\mathfrak{o}$.

In particular it holds that, for $\bar{m}_1, \bar{m}_2\in\mathfrak{o}_{ij}$ and $\bar{n}_1, \bar{n}_2\in\mathfrak{o}_{jk}$,

$$(\bar{m}_1\bar{m}_2) \cdot (\bar{n}_1\bar{n}_2) = (\bar{m}_1 \cdot \bar{n}_1)(\bar{m}_2 \cdot \bar{n}_2).$$

It is obvious that these operations make sense and $\mathfrak{o}_n^*(F, \mathfrak{q})$ is a ring under the operations. Throughout the rest of this paper we shall denote simply $\oplus$ by $+$, and skip $\otimes$ and $\cdot$ to avoid complicated notations.

Now we define $\hat{X}$ attached to $X=((f_{ij}x_{ij})^{-})$ in $\mathfrak{o}_n^*(F, \mathfrak{q})$ as in §1 by $\hat{X}=((f_{ij}x_{ji})^{-})$. This definition is independent of the choice of representatives $f_{ij}x_{ij} \bmod f_{ij}\mathfrak{q}$, and $\hat{X}$ is uniquely determined as an element of $\mathfrak{o}_n^*(F, \mathfrak{q})$.

Let $\mathfrak{o}_n(F, \mathfrak{q})$ be the set of matrices $M=\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)$ whose submatrices A, B, C, D belong to $\mathfrak{o}_n^*(F, \mathfrak{q})$, then the set forms a ring under the operations induced naturally from those in $\mathfrak{o}_n^*(F, \mathfrak{q})$. For any M in $\mathfrak{o}_n(F, \mathfrak{q})$ we define $\hat{M}$ by

$$\begin{pmatrix} \hat{A} & \hat{C} \\ \hat{B} & \hat{D}\end{pmatrix}.$$

Proposition 1. *Let us define* $\Delta(n, F, \mathfrak{q})=\{M \mid M\in\mathfrak{o}_n(F, \mathfrak{q})$ *and* $\hat{M}JM=J\}$[2] *for a fixed ideal* $\mathfrak{q}$ *of* $\mathfrak{o}$; *then* $\Delta(n, F, \mathfrak{q})$ *is a group under the same multiplication as* $\mathfrak{o}_n(F, \mathfrak{q})$.

Proof. As $[MN]^{\wedge}=\hat{N}\hat{M}$ for any M, N in $\mathfrak{o}_n(F, \mathfrak{q})$, $MN\in\Delta(n, F, \mathfrak{q})$ if $M, N\in\Delta(n, F, \mathfrak{q})$. The inverse element of $\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)$ is

$$\begin{pmatrix} \hat{D} & -\hat{B} \\ -\hat{C} & \hat{A}\end{pmatrix}.$$

Remark. $\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)$ is in $\Gamma(n, F)$ (or $\Delta(n, F, \mathfrak{q})$) if and only if $\hat{A}D-\hat{C}B=E$ and $\hat{A}C=\hat{C}A$, $\hat{B}D=\hat{D}B$ in $\mathfrak{o}_n^*(F)$ (or in $\mathfrak{o}_n^*(F, \mathfrak{q})$).

3. **Proof of Theorem 1.** Let ϕ_{ij} be a natural homomorphism of $f_{ij}\mathfrak{o}$ onto $\mathfrak{o}_{ij}$ and ϕ^* be a mapping of $\mathfrak{o}_n^*(F)$ such that for any $X=(f_{ij}x_{ij})\in\mathfrak{o}_n^*(F)$, $\phi^*(X)=(\phi_{ij}(f_{ij}x_{ij}))$. Let us define a mapping ϕ of $\mathfrak{o}_n(F)$ by

$$\phi(M) = \begin{pmatrix} \phi^*(A) & \phi^*(B) \\ \phi^*(C) & \phi^*(D)\end{pmatrix},$$

where $M=\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)$, and $A=(f_{ij}a_{ij}), \cdots$. We can easily check by considering §2 that ϕ is a homomorphism of $\mathfrak{o}_n(F)$ onto $\mathfrak{o}_n(F, \mathfrak{q})$, and ϕ induces a homomorphism of $\Gamma(n, F)$ into $\Delta(n, F, \mathfrak{q})$ whose kernel is $\Gamma(n, F, \mathfrak{q})$.

In the following we shall prove the ontoness of ϕ of $\Gamma(n, F)$ by using induction on n.

When $n=1$, then $F=I$, which leads to a well-known result (Klingen **[3]**, or Hurwitz **[2]**). So we assume that our proposition is true for any positive integer less than n. Let

$$\bar{M} = \begin{pmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D}\end{pmatrix}$$

[2] Except when confusion might arise, we use the same symbols for 1, E, J, etc. both in $\mathfrak{o}$ and $\mathfrak{o}_{ij}$ $(i\geqq j)$ or both in $\mathfrak{o}_n(F)$ $(\mathfrak{o}_n^*(F))$ and $\mathfrak{o}_n(F, \mathfrak{q})$ $(\mathfrak{o}_n^*(F, \mathfrak{q}))$.

be in $\Delta(n, F, \mathfrak{q})$, and let M be in $\mathfrak{o}_n(F)$ and be one of the inverse images of $\bar{M}$. By the remark of Proposition 1, $[\bar{A}]^{\wedge}\bar{D}-[\bar{C}]^{\wedge}\bar{B}=E$ holds and so the (1, 1) element of the left-hand side is 1,

$$\sum_{j=1}^{n}\{\phi_{11}(f_{1j}a_{j1}d_{j1})-\phi_{11}(f_{1j}c_{j1}b_{j1})\}=1.$$

Therefore

$$\langle a_{11}, f_{12}a_{21}, \cdots, f_{1n}a_{n1}, c_{11}, \cdots, f_{1n}c_{n1}, \mathfrak{q}\rangle = 1,$$

where $\langle\cdots\rangle$ means the largest common divisor. If $a_{21}=\cdots=a_{n1}=c_{11}=\cdots=c_{n1}=0$, we get

$$T_S M = \begin{pmatrix} a_{11} & \\ \cdot & \\ \cdot & \\ \cdot & \\ a_{11} & * \\ \cdot & \\ \cdot & \\ \cdot & \end{pmatrix} \quad \text{for } T_S = \begin{pmatrix} 1 & & & \\ & E_{n-1} & & \\ 1 & & 1 & \\ & & & E_{n-1} \end{pmatrix} \in \Gamma(n, F),$$

so we may assume that the set $(a_{21}, \cdots, a_{n1}, c_{11}, \cdots, c_{n1})$ contains at least a nonzero element. Then by Hilfssatz 1 of Klingen [3], there exists x in $\mathfrak{q}$, with $\langle a_{11}+x, f_{12}a_{21}, \cdots, f_{1n}a_{n1}, c_{11}, f_{12}c_{21}, \cdots, f_{1n}c_{n1}\rangle=1$, so that we can find $x_1, \cdots, x_n, y_1, \cdots, y_n$ in $\mathfrak{o}$ satisfying

$$x_1(a_{11}+x)+\sum_{i=2}^{n} x_i f_{1i} a_{i1} + \sum_{i=1}^{n} y_i f_{1i} c_{i1} = 1.$$

Since $\langle x_1, f_{12}x_2, \cdots, f_{1n}x_n, y_1, \cdots, f_{1n}y_n\rangle = 1$, therefore by Hilfssatz 3 of Christian [1][3] there exists a matrix N_1 in $\Gamma(n, F)$ whose first row is $(x_1, f_{12}x_2, \cdots, f_{1n}x_n, y_1, \cdots, f_{1n}y_n)$. For this N_1, $N_1 M \equiv \left(\begin{smallmatrix} 1 & * \\ * & * \end{smallmatrix}\right) \bmod (\mathfrak{q}, F)$.[4] Accordingly for a suitable matrix $N_2 \in \Gamma(n, F)$,

$$N_2 M \equiv \begin{pmatrix} 1 & * & * & * \\ & A_1 & * & B_1 \\ & & 1 & \\ & C_1 & * & D_1 \end{pmatrix} \bmod (\mathfrak{q}, F).$$

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[3] A very small modification is necessary because $\mathfrak{o}$ is the rational integer ring in this Hilfssatz. We can do it by using Hilfssatz 1, 2, 3 of Klingen [3].

[4] We use this notation in the following sense: "$L \equiv N \bmod (\mathfrak{q}, F)$ for $L, N \in \mathfrak{o}_n(F)$" is equivalent to "$\phi(L)=\phi(N)$".

Let $F_1=\{f_{ij}|f_{ij}\in F,\ 2\leqq i,j\leqq n\}$ and ϕ_1 be a homomorphism induced by ϕ of $\mathfrak{o}_{n-1}(F_1)$ onto $\mathfrak{o}_{n-1}(F_1,\mathfrak{q})$. Then

$$\phi_1\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$$

is in $\Delta(n-1, F_1, \mathfrak{q})$, since $\phi(N_2M)=\phi(N_2)\phi(M)$ is in $\mathfrak{o}_n(F,\mathfrak{q})$.

By our inductive assumption there exists a matrix N_3^* in $\Gamma(n-1, F_1)$ such that

$$\phi_1(N_3^*) = \phi_1\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}.$$

Thus we can find N_3 in $\Gamma(n, F)$ such that

$$N_3N_2M \equiv \begin{pmatrix} 1 & \mathfrak{a} & & \\ & & & S \\ & E_{n-1} & & \\ & & 1 & \\ & & \mathfrak{d} & E_{n-1} \end{pmatrix} \bmod (\mathfrak{q}, F),$$

where $\mathfrak{a}=(f_{12}a'_{12},\cdots,f_{1n}a'_{1n})$, $\mathfrak{d}=(d'_{21},\cdots,d'_{n1})$ and

$$S = \begin{pmatrix} b'_{11} & \cdots & f_{1n}b'_{1n} \\ \cdot & & \\ \cdot & 0 & \\ \cdot & & \\ b'_{n1} & & \end{pmatrix}$$

with $a'_{1i}+d'_{i1}\equiv 0 \bmod \mathfrak{q}$ and $b'_{1i}\equiv b'_{i1} \bmod \mathfrak{q}$.

We conclude that there exists a matrix N_4 in $\Gamma(n, F)$ such that

$$N_4N_3N_2M \equiv \begin{pmatrix} E & S^* \\ & E \end{pmatrix} = T_{S^*} \bmod (\mathfrak{q}, F),$$

where T_{S^*} is in $\Gamma(n, F)$ with $\hat{S}^*=S^*$.

Set $N_0=N_2^{-1}N_3^{-1}N_4^{-1}T_{S^*}$, then $N_0\in\Gamma(n, F)$ and $\bar{M}=\phi(M)=\phi(N_0)$. This shows that ϕ is a homomorphism of $\Gamma(n, F)$ onto $\Delta(n, F, \mathfrak{q})$, which completes our proof.

4. Proofs of Theorem 2 and its Corollary. To simplify our problem we prove

PROPOSITION 2. *If* $\mathfrak{q}=\mathfrak{q}_1\mathfrak{q}_2$ *with* $\langle\mathfrak{q}_1,\mathfrak{q}_2\rangle=1$, *then* $\Delta(n, F, \mathfrak{q})=\Delta(n, F, \mathfrak{q}_1)\otimes\Delta(n, F, \mathfrak{q}_2)$, *where* $\otimes$ *means the direct product of the groups.*

PROOF. For any $\phi(M)\in\Delta(n, F, \mathfrak{q})$ with $M\in\Gamma(n, F)$, let us associate a pair of matrices $M_k\in\mathfrak{o}_n(F)$ such that $M_k\equiv M \bmod(\mathfrak{q}_k, F)$, $k=1, 2$. This induces an isomorphism of $\Delta(n, F, \mathfrak{q})$ into $\Delta(n, F, \mathfrak{q}_1)\otimes\Delta(n, F, \mathfrak{q}_2)$. Conversely for any pair $\phi_k(M_k)\in\Delta(n, F, \mathfrak{q}_k)$ with $M_k\in\Gamma(n, F)$, $k=1, 2$, we determine $M\in\mathfrak{o}_n(F)$ such that $M_k\equiv M \bmod(\mathfrak{q}_k, F)$, $k=1, 2$. $M=q_1M_2+q_2M_1$ is a solution of the congruence equation, where $q_1+q_2=1$ with $q_k\in\mathfrak{q}_k$. For this M we have $\phi(\hat{M}JM)=J$. This shows that the isomorphism is onto and completes our proof.

COROLLARY. *If* $\mathfrak{q}=\prod_{\mathfrak{p}|\mathfrak{q}}\mathfrak{p}^e$, $\mathfrak{p}$ *is a prime ideal, then*

$$\Delta(n, F, \mathfrak{q}) = \prod_{\mathfrak{p}|\mathfrak{q}} \otimes\,\Delta(n, F, \mathfrak{p}^e).$$

By Theorem 1 and the Corollary of Proposition 2, $\mu(n, F, \mathfrak{q})=$the order of $\Delta(n, F, \mathfrak{q})=\prod_{\mathfrak{p}|\mathfrak{q}}\mu(n, F, \mathfrak{p}^e)$. Therefore we shall consider only $\mu(n, F, \mathfrak{p}^e)$ and $\Delta(n, F, \mathfrak{p}^e)$.

Let us assume that $\langle f_{1k}, \mathfrak{p}\rangle=1$ but $\langle f_{1k+1}, \mathfrak{p}\rangle\neq 1$. If the transposed vector of the first column of $M\in\Delta(n, F, \mathfrak{p}^e)$ is $(\bar{a}_{11}, \cdots, \bar{a}_{n1}, \bar{c}_{11}, \cdots, \bar{c}_{n1})$, then as already stated

$$\langle \bar{a}_{11}, \cdots, [f_{1n}a_{n1}]^-, \bar{c}_{11}, \cdots, [f_{1n}c_{n1}]^-, \mathfrak{p}/\mathfrak{p}^e\rangle = 1$$

with $[f_{1i}a_{i1}]^-\in\mathfrak{o}_{11}$, and by the above assumption this is equivalent to

$$\langle \bar{a}_{11}, \cdots, \bar{f}_{1k}\bar{a}_{k1}, \bar{c}_{11}, \cdots, \bar{f}_{1k}\bar{c}_{k1}, \mathfrak{p}/\mathfrak{p}^e\rangle = 1.$$

Assume (i) $\langle \bar{a}_{i1}, \mathfrak{p}/\mathfrak{p}^e\rangle\neq 1$ for $i\leqq m-1$, but $\langle \bar{a}_{m1}, \mathfrak{p}/\mathfrak{p}^e\rangle=1$ for $m\leqq k$, or (ii) $\langle \bar{a}_{i1}, \mathfrak{p}/\mathfrak{p}^e\rangle\neq 1$ for $1\leqq i\leqq k$, $\langle \bar{c}_{i1}, \mathfrak{p}/\mathfrak{p}^e\rangle\neq 1$ for $i\leqq m-1$, but $\langle \bar{c}_{m1}, \mathfrak{p}/\mathfrak{p}^e\rangle=1$ for $m\leqq k$.

Case (i). We determine $[f_{1m}x_m]^-\in\mathfrak{o}_{1m}$ with $[f_{1m}x_m]^-\bar{a}_{m1}=1$, and put

$$U_1=\begin{pmatrix} V_{11} & \\ & V_{12}\end{pmatrix};\quad V_{11}=\begin{pmatrix} 0 & & [f_{1m}x_m]^- & \\ & E_{m-2} & & \\ \bar{a}_{m1} & & 0 & \\ & & & E_{n-m}\end{pmatrix},\quad \hat{V}_{11}V_{12}=E_n.$$

$U_1\in\Delta(n, F, \mathfrak{p}^e)$ and $U_1M=\begin{pmatrix} 1 & * \\ * & *\end{pmatrix}$. With a suitable $L\in\Delta(n, F, \mathfrak{p}^e)$,

$$LU_1M=\begin{pmatrix} 1 & * & \mathfrak{b}_1 & \\ & A_1 & \mathfrak{b}_2 & B_1 \\ & & 1 & \\ & C_1 & \mathfrak{d}_1 & D_1\end{pmatrix};\qquad \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1\end{pmatrix}\in\Delta(n-1, F_1, \mathfrak{p}^e).$$

Moreover with U_2 and T_S of $\Delta(n, F, \mathfrak{p}^e)$, we have

$$LU_1MU_2T_S = \begin{pmatrix} 1 & & & \\ & A_1 & & B_1 \\ & & 1 & \\ & C_1 & & D_1 \end{pmatrix}, \quad \text{where } U_2 = \begin{pmatrix} V_{21} & \\ & V_{22} \end{pmatrix},$$

with

$$V_{21} = \begin{pmatrix} 1 & -\mathfrak{a} \\ & E_{n-1} \end{pmatrix}$$

and $\hat{V}_{21}V_{22}=E_n$, $T_S=\begin{pmatrix} E & S \\ & E \end{pmatrix}$ with

$$S = \begin{pmatrix} * & -\mathfrak{b}_1 \\ -\hat{\mathfrak{b}}_1 & 0 \end{pmatrix}.$$

Case (ii). We determine $[f_{1m}x_m]^{-}\in\mathfrak{o}_{1m}$ with $[f_{1m}x_m]^{-}\bar{c}_{m1}=1$, and put

$$U_1 = \begin{pmatrix} 0 & & & & & & & -[f_{1m}x_m]^{-} \\ & E_m & & & & & & \\ & & 0 & & -\bar{x}_m & & & \\ & & & E_{n-m-2} & & & & \\ & & [f_{1m}c_{m1}]^{-} & & 0 & & & \\ & & & & & E_m & & \\ \bar{c}_{m1} & & & & & & 0 & \\ & & & & & & & E_{n-m-2} \end{pmatrix}.$$

Then U_1 is in $\Delta(n, F, \mathfrak{p}^e)$ and $U_1M=\begin{pmatrix} 1 & * \\ * & * \end{pmatrix}$. So that, by the same method as in (i), we obtain the same result. In these transformations L and U_1 are unique, and $\mathfrak{a}$, $\mathfrak{b}_1$ are uniquely determined by given matrices

$$\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}, \quad \mathfrak{b} \text{ and } \mathfrak{b}_2.$$

So we have a recurrence formula

$$\mu(n, F, \mathfrak{p}^e) = N(\mathfrak{p}^{e(4n-1)})(1 - N(\mathfrak{p}^{-2k}))\mu(n-1, F_1, \mathfrak{p}^e),$$

where $N(\cdots)$ is the norm of the ideal. Considering the well-known relation (A. Hurwitz [**2**], or H. Klingen [**3**]), $\mu(1, 1, \mathfrak{p}^e)=N(\mathfrak{p}^{3e})(1-N(\mathfrak{p}^{-2}))$, we obtain

$$\mu(n, F, \mathfrak{p}^e) = N(\mathfrak{p}^{en(2n+1)}) \prod_{i=1}^{n} (1 - N(\mathfrak{p}^{-2k_i})),$$

where $k_i=k_i'-i+1$ is the largest integer with $\langle f_{ik_i'}, \mathfrak{p}\rangle=1$ for fixed i $(1\leqq i\leqq n)$.

Therefore for a general ideal $\mathfrak{q}$,

$$\mu(n, F, \mathfrak{q}) = N(\mathfrak{q})^{n(2n+1)} \prod_{\mathfrak{p}|\mathfrak{q}} \prod_{i=1}^{n} (1 - N(\mathfrak{p}^{-2k_{i,\mathfrak{p}}})),$$

where $k_{i,\mathfrak{p}}$ means the above explained k_i for $\mathfrak{p}$.

The same method as previously used can be applied to obtain the order $\nu(n, F, \mathfrak{q})$ of $\Gamma^0(n, F, \mathfrak{q})/\Gamma(n, F, \mathfrak{q})$. By Theorem 1, $\Gamma^0(n, F, \mathfrak{p}^e)/\Gamma(n, F, \mathfrak{p}^e)$ is isomorphic to a subgroup of $\Delta(n, F, \mathfrak{p}^e)$ consisting of matrices of such a form $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$. So with suitable L, $R \in \Delta(n, F, \mathfrak{p}^e)$ we obtain

$$LMR = \begin{pmatrix} 1 & & & \\ & A_1 & & B_1 \\ & & 1 & \\ & & & D_1 \end{pmatrix} \quad \text{with} \begin{pmatrix} A_1 & B_1 \\ 0 & D_1 \end{pmatrix} \in \Delta(n-1, F_1, \mathfrak{p}^e),$$

and so

$$\nu(n, F, \mathfrak{p}^e) = N(\mathfrak{p}^{e(3n-1)})(1 - N(\mathfrak{p}^{-k}))\nu(n-1, F_1, \mathfrak{p}^e).$$

Considering $\nu(1, 1, \mathfrak{p}^e) = N(\mathfrak{p}^{2e})(1 - N(\mathfrak{p}^{-1}))$, we have

$$\nu(n, F, \mathfrak{p}^e) = N(\mathfrak{p}^{en(3n+1)/2}) \prod_{i=1}^{n} (1 - N(\mathfrak{p}^{-k_i})).$$

So for a general ideal $\mathfrak{q}$,

$$\nu(n, F, \mathfrak{q}) = N(\mathfrak{q}^{n(3n+1)/2}) \prod_{\mathfrak{p}|\mathfrak{q}} \prod_{i=1}^{n} (1 - N(\mathfrak{p}^{-k_{i,\mathfrak{p}}})).$$

Therefore

$$\mu^0(n, F, \mathfrak{q}) = \mu(n, F, \mathfrak{q})/\nu(n, F, \mathfrak{q}) = N(\mathfrak{q}^{n(n+1)/2}) \prod_{\mathfrak{p}|\mathfrak{q}} \prod_{i=1}^{n} (1 + N(\mathfrak{p}^{-k_{i,\mathfrak{p}}})),$$

and we have proved Theorem 2.

Proof of the Corollary of Theorem 2. From the definition of $k_{i,\mathfrak{p}}$ it follows that $k_{i,\mathfrak{p}} = n - i + 1$, accordingly,

$$\mu(n, I, \mathfrak{q}) = \mu(n, F, \mathfrak{q}) = N(\mathfrak{q})^{n(2n+1)} \prod_{\mathfrak{p}|\mathfrak{q}} \prod_{i=1}^{n} (1 - N(\mathfrak{p}^{-2i})).$$

Let

$$T = \begin{pmatrix} 1 & & & & \\ & f_{12} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & f_{1n} \end{pmatrix}, \qquad T \in \mathfrak{o}_n^*(F),$$

then $\phi^*(T)\in\mathfrak{o}_n^*(F, \mathfrak{q})$ and $\phi_{ii}(f_{1i})$ is a unit in $\mathfrak{o}_{ii}$ for all i. Therefore $\phi\begin{pmatrix}T & \\ & E\end{pmatrix}\Delta(n, F, \mathfrak{q})\left(\phi\begin{pmatrix}T & \\ & E\end{pmatrix}\right)^{-1}$ is an isomorphic image in $\Delta(n, I, \mathfrak{q})$ of $\Delta(n, F, \mathfrak{q})$. Comparing the orders of both groups, it follows that $\Delta(n, F, \mathfrak{q})$ and $\Delta(n, I, \mathfrak{q})$ are isomorphic.

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