PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 32, Number 2, April 1972

AN INEQUALITY OF TURAN TYPE FOR JACOBI POLYNOMIALS

GEORGE GASPER

ABSTRACT. For Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, $\alpha, \beta > -1$, let

$$R_n(x) = \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)}, \qquad \Delta_n(x) = R_n^2(x) - R_{n-1}(x)R_{n+1}(x).$$

We prove that

$$\Delta_n(x) \geqq \frac{(\beta-\alpha)(1-x)}{2(n+\alpha+1)(n+\beta)} R_n^2(x), \qquad -1 \leqq x \leqq 1, n \geqq 1,$$

with equality only for $x = \pm 1$. This shows that the Turán inequality $\Delta_n(\alpha) \geqq 0$, $-1 \leqq x \leqq 1$, holds if and only if $\beta \geqq \alpha > -1$.

1. Let $R_n(x; \alpha, \beta) = P_n^{(\alpha,\beta)}(x)/P_n^{(\alpha,\beta)}(1)$ where $P_n^{(\alpha,\beta)}(x)$ is the Jacobi polynomial [8] of order (α, β), $\alpha, \beta > -1$, and let

$$\Delta_n(x; \alpha, \beta) = R_n^2(x; \alpha, \beta) - R_{n-1}(x; \alpha, \beta)R_{n+1}(x; \alpha, \beta).$$

In [3, p. 153] Karlin and Szegö raised the question of whether or not the Turán inequality

$$\Delta_n(x; \alpha, \beta) > 0, \qquad -1 < x < 1, n \geqq 1, \tag{1}$$

holds for $\beta \geqq \alpha > -1$. From

$$\Delta_n(-1; \alpha, \beta) = \frac{(\beta-\alpha)}{(n+\alpha+1)(n+\beta)} \binom{n+\beta}{n}^2 \binom{n+\alpha}{n}^{-2}$$

it is obvious that (1) fails for $\beta < \alpha$. Later [7] Szegö showed that (1) holds for $\beta \geqq |\alpha|$, $\alpha > -1$, but unfortunately his method fails for the triangle

$$U = \{(\alpha, \beta): \alpha < \beta < -\alpha, -1 < \alpha < 0\}.$$

Recently the author [2] proved (1) for the set

$$V = \{(\alpha, \beta): \beta \geqq \alpha > -1, (\beta-\alpha)(\alpha+\beta)(4\beta^2+4\alpha+1) \geqq 0\},$$

Received by the editors July 14, 1971.

AMS 1970 *subject classifications*. Primary 33A50, 33A65, 42A04; Secondary 33A10, 42A52.

Key words and phrases. Inequality, Turán inequality, Jacobi polynomials, ultraspherical polynomials, orthogonal polynomials, Dirichlet kernel.

which contains most of U, but again the method employed failed for the set $U - V$. In this note we give a proof of (1) for the "best possible" range $\beta \geqq \alpha > -1$ and also establish the following stronger result.

THEOREM. *Let $\alpha > -1$, $\beta > -1$ and $n \geqq 1$. Then*

$$\Delta_n(x;\alpha,\beta) \geqq \frac{(\beta-\alpha)(1-x)}{2(n+\alpha+1)(n+\beta)} R_n^2(x;\alpha,\beta), \qquad -1 \leqq x \leqq 1, \tag{2}$$

with equality only for $x = \pm 1$.

The method used to prove (2) is a modification of that used by Mukherjee and Nanjundiah [**4**] for Laguerre and Hermite polynomials, and by Skovgaard [**6**] for ultraspherical polynomials and Bessel functions. It depends on the observation that if $q_n(x)$ is a polynomial of degree n with simple zeros $x_1, x_2, \cdots, x_n$, then

$$\frac{q_n'}{q_n} = \sum_{k=1}^{n} (x - x_k)^{-1},$$

$$-\left(\frac{q_n'}{q_n}\right)' = \frac{q_n'^2 - q_n q_n''}{q_n^2} = \sum_{k=1}^{n} (x - x_k)^{-2},$$

where a prime denotes differentiation with respect to x. These identities were used almost a hundred years ago by Laguerre (see [**8**, p. 120]) to derive bounds for the zeros of the classical polynomials.

Before proving (2), let us first point out some interesting special cases. Using $P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x)$, we find that (2) may be rewritten in the form

$$\begin{aligned} &\frac{(n+\alpha)(n+\beta+1)}{(n+\alpha+1)(n+\beta)} R_n^2(x;\alpha,\beta) - R_{n-1}(x;\alpha,\beta)R_{n+1}(x;\alpha,\beta) \\ &\qquad \geqq \frac{(\alpha-\beta)(1+x)}{2(n+\alpha+1)(n+\beta)} R_n^2(x;\alpha,\beta), \qquad -1 \leqq x \leqq 1, \end{aligned} \tag{3}$$

where equality holds only for $x = \pm 1$. Since

$$R_n(\cos\theta; \tfrac{1}{2}, -\tfrac{1}{2}) = \frac{\sin(n+\frac{1}{2})\theta}{(2n+1)\sin(\theta/2)}$$

[**8**, p. 60], it follows from (3) that the Dirichlet kernel

$$D_n(\theta) = \tfrac{1}{2} + \cos\theta + \cdots + \cos n\theta = \frac{\sin(n+\frac{1}{2})\theta}{2\sin(\theta/2)}$$

satisfies

$$D_n^2(\theta) - D_{n-1}(\theta)D_{n+1}(\theta) \geqq \frac{2(1+\cos\theta)}{(2n+1)^2} D_n^2(\theta),$$

with equality only when $\cos\theta = \pm 1$.

If we let $p_n(x;\lambda) = P_n^{(\lambda)}(x)/P_n^{(\lambda)}(1)$ where $P_n^{(\lambda)}(x)$ is the ultraspherical polynomial [**8**, p. 81] of order λ, $\lambda > -\frac{1}{2}$, then from [**8**, p. 59] we have

$$p_{2n}(x;\lambda) = R_n(2x^2-1;\lambda-\tfrac{1}{2},-\tfrac{1}{2}),$$
$$p_{2n+1}(x;\lambda) = xR_n(2x^2-1;\lambda-\tfrac{1}{2},\tfrac{1}{2}).$$

Hence (2) and (3) yield, for $\lambda > -\frac{1}{2}$ and $-1 \leqq x \leqq 1$,

$$p_{2n}^2(x;\lambda) - p_{2n-2}(x;\lambda)p_{2n+2}(x;\lambda) \geqq -\frac{4\lambda(1-x^2)}{(2n-1)(2n+2\lambda+1)} p_{2n}^2(x;\lambda),$$

$$p_{2n+1}^2(x;\lambda) - p_{2n-1}(x;\lambda)p_{2n+3}(x;\lambda) \geqq \frac{4(1-\lambda)(1-x^2)}{(2n+1)(2n+2\lambda+1)} p_{2n+1}^2(x;\lambda),$$

$$\frac{(2n+1)(2n+2\lambda-1)}{(2n-1)(2n+2\lambda+1)} p_{2n}^2(x;\lambda) - p_{2n-2}(x;\lambda)p_{2n+2}(x;\lambda) \geqq \frac{4\lambda x^2}{(2n-1)(2n+2\lambda+1)} p_{2n}^2(x;\lambda),$$

$$\frac{(2n+3)(2n+2\lambda-1)}{(2n+1)(2n+2\lambda+1)} p_{2n+1}^2(x;\lambda) - p_{2n-1}(x;\lambda)p_{2n+3}(x;\lambda) \geqq \frac{4(\lambda-1)x^2}{(2n+1)(2n+2\lambda+1)} p_{2n+1}^2(x;\lambda),$$

with equality only for $x = -1, 0, 1$. This sharpens the inequalities in [**2**, Corollaries 1 and 2]. For other types of inequalities for orthogonal polynomials see Askey [**1**] and Patrick [**5**].

2. In proving the theorem we may assume that $-1 < x < 1$, for from

$$R_n(1;\alpha,\beta) = 1, \qquad R_n(-1;\alpha,\beta) = (-1)^n \binom{n+\beta}{n}\binom{n+\alpha}{n}^{-1}$$

it is easy to see that equality holds in (2) when $x = \pm 1$. Since consecutive orthogonal polynomials $q_n(x)$ and $q_{n+1}(x)$ cannot have common zeros [**8**, §3.3], we may also assume that x is not a zero of $R_n(x;\alpha,\beta)$. Fix $n \geqq 1$, $\alpha > -1$, $\beta > -1$ and let $R_n = R_n(x;\alpha,\beta)$, $\Delta_n = \Delta_n(x;\alpha,\beta)$. From [**8**, p. 72] we have

$$R_{n-1} = A_nR_n' + B_nR_n, \qquad R_{n+1} = C_nR_n' + D_nR_n,$$

where A_n, B_n, C_n, D_n are functions of x defined by

$$A_n = \frac{2n+\alpha+\beta}{2n(n+\beta)}(1-x^2), \qquad B_n = \frac{2n+\alpha+\beta}{2(n+\beta)}\left(x + \frac{\beta-\alpha}{2n+\alpha+\beta}\right),$$

$$C_n = \frac{2n+\alpha+\beta+2}{2(n+\alpha+1)(n+\alpha+\beta+1)}(x^2-1),$$

$$D_n = \frac{2n+\alpha+\beta+2}{2(n+\alpha+1)}\left(x + \frac{\alpha-\beta}{2n+\alpha+\beta+2}\right).$$

Using

$$(R_n'/R_n)' = (R_nR_n'' - R_n'^2)/R_n^2$$

and the differential equation [**8**, p. 60] satisfied by R_n we obtain

$$\begin{aligned}(1-x^2)R_n'^2 = {} & [\alpha-\beta+(\alpha+\beta+2)x]R_nR_n' \\ & - n(n+\alpha+\beta+1)R_n^2 - (1-x^2)(R_n'/R_n)'R_n^2.\end{aligned}$$

Hence

$$\begin{aligned}\Delta_n &= R_n^2 - (A_nR_n' + B_nR_n)(C_nR_n' + D_nR_n) \\ &= [1 - B_nD_n + n(n+\alpha+\beta+1)(1-x^2)^{-1}A_nC_n]R_n^2 \\ &\quad + [-A_nD_n - B_nC_n + (1-x^2)^{-1}(\beta-\alpha-(\alpha+\beta+2)x)A_nC_n]R_nR_n' \\ &\quad + (R_n'/R_n)'A_nC_nR_n^2 \\ &= \frac{(\beta-\alpha)(1-x)}{2(n+\alpha+1)(n+\beta)}R_n^2 \\ &\quad - \left[\left(x + \frac{\alpha^2-\beta^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}\right)\frac{R_n'}{R_n} - (1-x^2)\left(\frac{R_n'}{R_n}\right)'\right] \\ &\quad \times \frac{A_nC_nR_n^2}{1-x^2}.\end{aligned}$$

Now recall [**8**, §3.3] that all zeros of $R_n(x;\alpha,\beta)$ are real and simple and are located in the open interval $(-1, 1)$. Thus

$$\frac{R_n'}{R_n} = \sum_{k=1}^{n}(x-x_k)^{-1}, \qquad \left(\frac{R_n'}{R_n}\right)' = -\sum_{k=1}^{n}(x-x_k)^{-2},$$

where $x_1, x_2, \cdots, x_n$ are the zeros of R_n, and so

$$\begin{aligned}\Delta_n = {} & \frac{(\beta-\alpha)(1-x)}{2(n+\alpha+1)(n+\beta)}R_n^2 \\ & - \frac{(1-x^2)^{-1}A_nC_nR_n^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}\sum_{k=1}^{n}\frac{E(k,n;\alpha,\beta)}{(x-x_k)^2}\end{aligned} \tag{4}$$

with

$$E(k, n; \alpha, \beta) = (2n + \alpha + \beta)(2n + \alpha + \beta + 2)(1 - xx_k) + (\alpha^2 - \beta^2)(x - x_k).$$

To complete the proof it suffices to observe that the expression $-A_nC_nE(k, n; \alpha, \beta)$ in (4) is (strictly) positive since $-A_nC_n>0$ and

$$\begin{aligned} E(k, n; \alpha, \beta) &= 2[2(n-1)(n+\alpha+\beta+2) + 2(\alpha+1) \\ &\qquad + (\beta+1)(\alpha+\beta+2)](1 - xx_k) \\ &\quad + (\alpha^2 - \beta^2)(1 + x)(1 - x_k) \\ &= 2[2(n-1)(n+\alpha+\beta+2) + 2(\beta+1) \\ &\qquad + (\alpha+1)(\alpha+\beta+2)](1 - xx_k) \\ &\quad + (\beta^2 - \alpha^2)(1 - x)(1 + x_k) > 0 \end{aligned}$$

for $-1<x, x_k<1$.

References

1. R. Askey, *An inequality for the classical polynomials*, Nederl. Akad. Wetensch. Proc. Ser. A **73**=Indag. Math. **32** (1970), 22–25. MR **41** #2079.

2. G. Gasper, *On the extension of Turán's inequality to Jacobi polynomials*, Duke Math. J. **38** (1971), 415–428.

3. S. Karlin and G. Szegö, *On certain determinants whose elements are orthogonal polynomials*, J. Analyse Math. **8** (1960/61), 1–157. MR **26** #539.

4. B. N. Mukherjee and T. S. Nanjundiah, *On an inequality relating to Laguerre and Hermite polynomials*, Math. Student **19** (1951), 47–48. MR **13**, 649.

5. M. L. Patrick, *Some inequalities concerning Jacobi polynomials*, SIAM J. Math. Anal. **2** (1971), 213–220.

6. H. Skovgaard, *On inequalities of the Turán type*, Math. Scand. **2** (1954), 65–73. MR **16**, 118.

7. G. Szegö, *An inequality for Jacobi polynomials*, Studies in Math. Anal. and Related Topics, Stanford Univ. Press, Stanford, Calif., 1962, pp. 392–398. MR **26** #2657.

8. ———, *Orthogonal polynomials*, rev. ed., Amer. Math. Soc. Colloq. Publ., vol. 23, Amer. Math. Soc., Providence, R.I., 1959. MR **21** #5029.

Department of Mathematics, Northwestern University, Evanston, Illinois 60201