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AN INEQUALITY OF TURAN TYPE FOR JACOBI POLYNOMIALS

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ABSTRACT. For Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, $\alpha, \beta > -1$, let

$$R_n(x) = \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)}, \qquad \Delta_n(x) = R_n^2(x) - R_{n-1}(x)R_{n+1}(x).$$

We prove that

$$\Delta_n(x) \geqq \frac{(\beta - \alpha)(1 - x)}{2(n + \alpha + 1)(n + \beta)} R_n^2(x), \qquad -1 \leqq x \leqq 1, n \geqq 1,$$

with equality only for $x = \pm 1$. This shows that the Turán inequality $\Delta_n(\alpha) \geqq 0$, $-1 \leqq x \leqq 1$, holds if and only if $\beta \geqq \alpha > -1$.

1. Let $R_n(x; \alpha, \beta) = P_n^{(\alpha,\beta)}(x)/P_n^{(\alpha,\beta)}(1)$ where $P_n^{(\alpha,\beta)}(x)$ is the Jacobi polynomial [8] of order (α, β), $\alpha, \beta > -1$, and let

$$\Delta_n(x; \alpha, \beta) = R_n^2(x; \alpha, \beta) - R_{n-1}(x; \alpha, \beta)R_{n+1}(x; \alpha, \beta).$$

In [3, p. 153] Karlin and Szegö raised the question of whether or not the Turán inequality

$$\Delta_n(x; \alpha, \beta) > 0, \qquad -1 < x < 1, n \geqq 1, \tag{1}$$

holds for $\beta \geqq \alpha > -1$. From

$$\Delta_n(-1; \alpha, \beta) = \frac{(\beta - \alpha)}{(n + \alpha + 1)(n + \beta)} \binom{n + \beta}{n}^2 \binom{n + \alpha}{n}^{-2}$$

it is obvious that (1) fails for $\beta < \alpha$. Later [7] Szegö showed that (1) holds for $\beta \geqq |\alpha|$, $\alpha > -1$, but unfortunately his method fails for the triangle

$$U = \{(\alpha, \beta): \alpha < \beta < -\alpha, -1 < \alpha < 0\}.$$

Recently the author [2] proved (1) for the set

$$V = \{(\alpha, \beta): \beta \geqq \alpha > -1, (\beta - \alpha)(\alpha + \beta)(4\beta^2 + 4\alpha + 1) \geqq 0\},$$

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which contains most of U, but again the method employed failed for the set $U - V$. In this note we give a proof of (1) for the "best possible" range $\beta \geqq \alpha > -1$ and also establish the following stronger result.

THEOREM. *Let $\alpha > -1$, $\beta > -1$ and $n \geqq 1$. Then*

$$\Delta_n(x;\alpha,\beta) \geqq \frac{(\beta-\alpha)(1-x)}{2(n+\alpha+1)(n+\beta)} R_n^2(x;\alpha,\beta), \qquad -1 \leqq x \leqq 1, \tag{2}$$

with equality only for $x = \pm 1$.

The method used to prove (2) is a modification of that used by Mukherjee and Nanjundiah [**4**] for Laguerre and Hermite polynomials, and by Skovgaard [**6**] for ultraspherical polynomials and Bessel functions. It depends on the observation that if $q_n(x)$ is a polynomial of degree n with simple zeros $x_1, x_2, \cdots, x_n$, then

$$\frac{q_n'}{q_n} = \sum_{k=1}^{n} (x - x_k)^{-1},$$

$$-\left(\frac{q_n'}{q_n}\right)' = \frac{q_n'^2 - q_n q_n''}{q_n^2} = \sum_{k=1}^{n} (x - x_k)^{-2},$$

where a prime denotes differentiation with respect to x. These identities were used almost a hundred years ago by Laguerre (see [**8**, p. 120]) to derive bounds for the zeros of the classical polynomials.

Before proving (2), let us first point out some interesting special cases. Using $P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x)$, we find that (2) may be rewritten in the form

$$\begin{aligned} &\frac{(n+\alpha)(n+\beta+1)}{(n+\alpha+1)(n+\beta)} R_n^2(x;\alpha,\beta) - R_{n-1}(x;\alpha,\beta) R_{n+1}(x;\alpha,\beta) \\ &\qquad \geqq \frac{(\alpha-\beta)(1+x)}{2(n+\alpha+1)(n+\beta)} R_n^2(x;\alpha,\beta), \qquad -1 \leqq x \leqq 1, \end{aligned} \tag{3}$$

where equality holds only for $x = \pm 1$. Since

$$R_n(\cos\theta; \tfrac{1}{2}, -\tfrac{1}{2}) = \frac{\sin(n+\frac{1}{2})\theta}{(2n+1)\sin(\theta/2)}$$

[**8**, p. 60], it follows from (3) that the Dirichlet kernel

$$D_n(\theta) = \tfrac{1}{2} + \cos\theta + \cdots + \cos n\theta = \frac{\sin(n+\frac{1}{2})\theta}{2\sin(\theta/2)}$$

satisfies

$$D_n^2(\theta) - D_{n-1}(\theta)D_{n+1}(\theta) \geqq \frac{2(1+\cos\theta)}{(2n+1)^2} D_n^2(\theta),$$

with equality only when $\cos\theta = \pm 1$.

If we let $p_n(x;\lambda) = P_n^{(\lambda)}(x)/P_n^{(\lambda)}(1)$ where $P_n^{(\lambda)}(x)$ is the ultraspherical polynomial [**8**, p. 81] of order λ, $\lambda > -\frac{1}{2}$, then from [**8**, p. 59] we have

$$p_{2n}(x;\lambda) = R_n(2x^2-1;\lambda-\tfrac{1}{2},-\tfrac{1}{2}),$$
$$p_{2n+1}(x;\lambda) = xR_n(2x^2-1;\lambda-\tfrac{1}{2},\tfrac{1}{2}).$$

Hence (2) and (3) yield, for $\lambda > -\frac{1}{2}$ and $-1 \leqq x \leqq 1$,

$$p_{2n}^2(x;\lambda) - p_{2n-2}(x;\lambda)p_{2n+2}(x;\lambda) \geqq -\frac{4\lambda(1-x^2)}{(2n-1)(2n+2\lambda+1)} p_{2n}^2(x;\lambda),$$

$$p_{2n+1}^2(x;\lambda) - p_{2n-1}(x;\lambda)p_{2n+3}(x;\lambda) \geqq \frac{4(1-\lambda)(1-x^2)}{(2n+1)(2n+2\lambda+1)} p_{2n+1}^2(x;\lambda),$$

$$\frac{(2n+1)(2n+2\lambda-1)}{(2n-1)(2n+2\lambda+1)} p_{2n}^2(x;\lambda) - p_{2n-2}(x;\lambda)p_{2n+2}(x;\lambda) \geqq \frac{4\lambda x^2}{(2n-1)(2n+2\lambda+1)} p_{2n}^2(x;\lambda),$$

$$\frac{(2n+3)(2n+2\lambda-1)}{(2n+1)(2n+2\lambda+1)} p_{2n+1}^2(x;\lambda) - p_{2n-1}(x;\lambda)p_{2n+3}(x;\lambda) \geqq \frac{4(\lambda-1)x^2}{(2n+1)(2n+2\lambda+1)} p_{2n+1}^2(x;\lambda),$$

with equality only for $x = -1, 0, 1$. This sharpens the inequalities in [**2**, Corollaries 1 and 2]. For other types of inequalities for orthogonal polynomials see Askey [**1**] and Patrick [**5**].

2. In proving the theorem we may assume that $-1 < x < 1$, for from

$$R_n(1;\alpha,\beta) = 1, \qquad R_n(-1;\alpha,\beta) = (-1)^n \binom{n+\beta}{n}\binom{n+\alpha}{n}^{-1}$$

it is easy to see that equality holds in (2) when $x = \pm 1$. Since consecutive orthogonal polynomials $q_n(x)$ and $q_{n+1}(x)$ cannot have common zeros [**8**, §3.3], we may also assume that x is not a zero of $R_n(x;\alpha,\beta)$. Fix $n \geqq 1$, $\alpha > -1$, $\beta > -1$ and let $R_n = R_n(x;\alpha,\beta)$, $\Delta_n = \Delta_n(x;\alpha,\beta)$. From [**8**, p. 72] we have

$$R_{n-1} = A_n R_n' + B_n R_n, \qquad R_{n+1} = C_n R_n' + D_n R_n,$$

where A_n, B_n, C_n, D_n are functions of x defined by

$$A_n = \frac{2n+\alpha+\beta}{2n(n+\beta)}(1-x^2), \qquad B_n = \frac{2n+\alpha+\beta}{2(n+\beta)}\left(x + \frac{\beta-\alpha}{2n+\alpha+\beta}\right),$$

$$C_n = \frac{2n+\alpha+\beta+2}{2(n+\alpha+1)(n+\alpha+\beta+1)}(x^2-1),$$

$$D_n = \frac{2n+\alpha+\beta+2}{2(n+\alpha+1)}\left(x + \frac{\alpha-\beta}{2n+\alpha+\beta+2}\right).$$

Using

$$(R_n'/R_n)' = (R_nR_n'' - R_n'^2)/R_n^2$$

and the differential equation [**8**, p. 60] satisfied by R_n we obtain

$$\begin{aligned}(1-x^2)R_n'^2 = {} & [\alpha-\beta+(\alpha+\beta+2)x]R_nR_n' \\ & - n(n+\alpha+\beta+1)R_n^2 - (1-x^2)(R_n'/R_n)'R_n^2.\end{aligned}$$

Hence

$$\begin{aligned}\Delta_n &= R_n^2 - (A_nR_n' + B_nR_n)(C_nR_n' + D_nR_n) \\ &= [1 - B_nD_n + n(n+\alpha+\beta+1)(1-x^2)^{-1}A_nC_n]R_n^2 \\ &\quad + [-A_nD_n - B_nC_n + (1-x^2)^{-1}(\beta-\alpha-(\alpha+\beta+2)x)A_nC_n]R_nR_n' \\ &\quad + (R_n'/R_n)'A_nC_nR_n^2 \\ &= \frac{(\beta-\alpha)(1-x)}{2(n+\alpha+1)(n+\beta)}R_n^2 \\ &\quad - \left[\left(x + \frac{\alpha^2-\beta^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}\right)\frac{R_n'}{R_n} - (1-x^2)\left(\frac{R_n'}{R_n}\right)'\right] \\ &\quad \times \frac{A_nC_nR_n^2}{1-x^2}.\end{aligned}$$

Now recall [**8**, §3.3] that all zeros of $R_n(x;\alpha,\beta)$ are real and simple and are located in the open interval $(-1, 1)$. Thus

$$\frac{R_n'}{R_n} = \sum_{k=1}^{n}(x-x_k)^{-1}, \qquad \left(\frac{R_n'}{R_n}\right)' = -\sum_{k=1}^{n}(x-x_k)^{-2},$$

where $x_1, x_2, \cdots, x_n$ are the zeros of R_n, and so

$$\begin{aligned}\Delta_n &= \frac{(\beta-\alpha)(1-x)}{2(n+\alpha+1)(n+\beta)}R_n^2 \\ &\quad - \frac{(1-x^2)^{-1}A_nC_nR_n^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}\sum_{k=1}^{n}\frac{E(k,n;\alpha,\beta)}{(x-x_k)^2}\end{aligned} \tag{4}$$

with

$$E(k, n; \alpha, \beta) = (2n + \alpha + \beta)(2n + \alpha + \beta + 2)(1 - xx_k) + (\alpha^2 - \beta^2)(x - x_k).$$

To complete the proof it suffices to observe that the expression $-A_nC_nE(k, n; \alpha, \beta)$ in (4) is (strictly) positive since $-A_nC_n > 0$ and

$$\begin{aligned} E(k, n; \alpha, \beta) &= 2[2(n-1)(n+\alpha+\beta+2) + 2(\alpha+1) \\ &\qquad + (\beta+1)(\alpha+\beta+2)](1 - xx_k) \\ &\quad + (\alpha^2 - \beta^2)(1+x)(1-x_k) \\ &= 2[2(n-1)(n+\alpha+\beta+2) + 2(\beta+1) \\ &\qquad + (\alpha+1)(\alpha+\beta+2)](1 - xx_k) \\ &\quad + (\beta^2 - \alpha^2)(1-x)(1+x_k) > 0 \end{aligned}$$

for $-1 < x, x_k < 1$.

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