AN INEQUALITY OF TURAN TYPE FOR
JACOBI POLYNOMIALS

GEORGE GASPER

Abstract. For Jacobi polynomials \( P_n^{(\alpha, \beta)}(x) \), \( \alpha, \beta > -1 \), let

\[
R_n(x) = \frac{P_n^{(\alpha, \beta)}(x)}{P_n^{(\alpha, \beta)}(1)}, \quad \Delta_n(x) = R_n^2(x) - R_n(x)R_{n+1}(x).
\]

We prove that

\[
2(n + \alpha + 1)(n + \beta)
\]

with equality only for \( x = \pm 1 \). This shows that the Turán inequality

\[
\Delta_n(x) > 0, \quad -1 < x < 1, \quad n \geq 1,
\]

holds if and only if \( \beta \geq \alpha > -1 \).

1. Let \( R_n(x, \alpha, \beta) = P_n^{(\alpha, \beta)}(x)/P_n^{(\alpha, \beta)}(1) \) where \( P_n^{(\alpha, \beta)}(x) \) is the Jacobi polynomial \( [8] \) of order \( (\alpha, \beta) \), \( \alpha, \beta > -1 \), and let

\[
\Delta_n(x; \alpha, \beta) = R_n^2(x; \alpha, \beta) - R_{n-1}(x; \alpha, \beta)R_{n+1}(x; \alpha, \beta).
\]

In \([3, p. 153]\) Karlin and Szegö raised the question of whether or not the Turán inequality

\[
\Delta_n(x; \alpha, \beta) > 0, \quad -1 < x < 1, \quad n \geq 1,
\]

holds for \( \beta \geq \alpha > -1 \). From

\[
\Delta_n(-1; \alpha, \beta) = \frac{(\beta - \alpha)}{(n + \alpha + 1)(n + \beta)} \left( \frac{n + \beta}{n} \right)^2 \left( \frac{n + \alpha}{n} \right)^{-2}
\]

it is obvious that (1) fails for \( \beta < \alpha \). Later \([7]\) Szegö showed that (1) holds for \( \beta \geq |\alpha|, \alpha > -1 \), but unfortunately his method fails for the triangle

\[
U = \{(\alpha, \beta) : \alpha < \beta < -\alpha, -1 < \alpha < 0\}.
\]

Recently the author \([2]\) proved (1) for the set

\[
V = \{(\alpha, \beta) : \beta \geq \alpha > -1, (\beta - \alpha)(\alpha + \beta)(4\beta^2 + 4\alpha + 1) \geq 0\},
\]
which contains most of $U$, but again the method employed failed for the set $U - V$. In this note we give a proof of (1) for the “best possible” range $\beta \geq \alpha > -1$ and also establish the following stronger result.

**Theorem.** Let $\alpha > -1$, $\beta > -1$ and $n \geq 1$. Then

$$\Delta_n(x; \alpha, \beta) \geq \frac{(\beta - \alpha)(1 - x)}{2(n + \alpha + 1)(n + \beta)} R_n^2(x; \alpha, \beta), \quad -1 \leq x \leq 1,$$

with equality only for $x = \pm 1$.

The method used to prove (2) is a modification of that used by Mukherjee and Nanjundiah [4] for Laguerre and Hermite polynomials, and by Skovgaard [6] for ultraspherical polynomials and Bessel functions. It depends on the observation that if $q_n(x)$ is a polynomial of degree $n$ with simple zeros $x_1, x_2, \cdots, x_n$, then

$$q_n' = \sum_{k=1}^{n} (x - x_k)^{-1},$$

$$-\left(\frac{q_n'}{q_n}\right)' = \frac{q_n'^2 - q_n q_n''}{q_n^2} = \sum_{k=1}^{n} (x - x_k)^{-2},$$

where a prime denotes differentiation with respect to $x$. These identities were used almost a hundred years ago by Laguerre (see [8, p. 120]) to derive bounds for the zeros of the classical polynomials.

Before proving (2), let us first point out some interesting special cases. Using $P_n^{(\alpha, \beta)}(-x) = (-1)^n P_n^{(\beta, \alpha)}(x)$, we find that (2) may be rewritten in the form

$$\frac{(n + \alpha)(n + \beta + 1)}{(n + \alpha + 1)(n + \beta)} R_n^2(x; \alpha, \beta) = R_{n-1}(x; \alpha, \beta) R_{n+1}(x; \alpha, \beta) \geq \frac{(\alpha - \beta)(1 + x)}{2(n + \alpha + 1)(n + \beta)} R_n^2(x; \alpha, \beta), \quad -1 \leq x \leq 1,$$

where equality holds only for $x = \pm 1$. Since

$$R_n(\cos \theta; \frac{1}{2}, -\frac{1}{2}) = \frac{\sin(n + \frac{1}{2})\theta}{(2n + 1)\sin(\theta/2)}$$

[8, p. 60], it follows from (3) that the Dirichlet kernel

$$D_n(\theta) = \frac{1}{2} + \cos \theta + \cdots + \cos n\theta = \frac{\sin(n + \frac{1}{2})\theta}{2 \sin(\theta/2)}$$
satisfies
\[ D_n^2(\theta) - D_{n-1}(\theta)D_{n+1}(\theta) \geq \frac{2(1 + \cos \theta)}{(2n + 1)^2} D_n^2(\theta), \]
with equality only when \( \cos \theta = \pm 1 \).

If we let \( p_n(x; \lambda) = \frac{P_n^{(\lambda)}(x)}{P_n^{(\lambda)}(1)} \) where \( P_n^{(\lambda)}(x) \) is the ultraspherical polynomial [8, p. 81] of order \( \lambda \), \( \lambda > -\frac{1}{2} \), then from [8, p. 59] we have
\begin{align*}
p_{2n}(x; \lambda) &= R_n(2x^2 - 1; \lambda - \frac{1}{2}, -\frac{1}{2}), \\
p_{2n+1}(x; \lambda) &= xR_n(2x^2 - 1; \lambda - \frac{1}{2}, \frac{1}{2}).
\end{align*}
Hence (2) and (3) yield, for \( \lambda > -\frac{1}{2} \) and \(-1 \leq x \leq 1\),
\begin{align*}
p_{2n}^2(x; \lambda) - p_{2n-2}(x; \lambda)p_{2n+2}(x; \lambda) &\geq -\frac{4\lambda(1 - x^2)}{(2n - 1)(2n + 2\lambda + 1)} p_{2n}^2(x; \lambda), \\
p_{2n+1}^2(x; \lambda) - p_{2n-1}(x; \lambda)p_{2n+3}(x; \lambda) &\geq -\frac{4(1 - \lambda)(1 - x^2)}{(2n + 1)(2n + 2\lambda + 1)} p_{2n+1}^2(x; \lambda), \\
\frac{(2n + 1)(2n + 2\lambda - 1)}{(2n - 1)(2n + 2\lambda + 1)} p_{2n}(x; \lambda) - p_{2n-2}(x; \lambda)p_{2n+2}(x; \lambda) &\geq -\frac{4\lambda x^2}{(2n - 1)(2n + 2\lambda + 1)} p_{2n}^2(x; \lambda), \\
\frac{(2n + 3)(2n + 2\lambda - 1)}{(2n + 1)(2n + 2\lambda + 1)} p_{2n+1}(x; \lambda) - p_{2n-1}(x; \lambda)p_{2n+3}(x; \lambda) &\geq -\frac{4(\lambda - 1)x^2}{(2n + 1)(2n + 2\lambda + 1)} p_{2n+1}^2(x; \lambda),
\end{align*}
with equality only for \( x = -1, 0, 1 \). This sharpens the inequalities in [2, Corollaries 1 and 2]. For other types of inequalities for orthogonal polynomials see Askey [1] and Patrick [5].

2. In proving the theorem we may assume that \(-1 < x < 1\), for from
\[ R_n(1; \alpha, \beta) = 1, \quad R_n(-1; \alpha, \beta) = (-1)^n \binom{n + \beta}{n} \binom{n + \alpha}{n}^{-1} \]
it is easy to see that equality holds in (2) when \( x = \pm 1 \). Since consecutive orthogonal polynomials \( q_n(x) \) and \( q_{n+1}(x) \) cannot have common zeros [8, §3.3], we may also assume that \( x \) is not a zero of \( R_n(x; \alpha, \beta) \). Fix \( n \geq 1, \alpha > -1, \beta > -1 \) and let \( R_n = R_n(x; \alpha, \beta), \Delta_n = \Delta_n(x; \alpha, \beta) \). From [8, p. 72] we have
\[ R_{n-1} = A_nR'_n + B_nR_n, \quad R_{n+1} = C_nR'_n + D_nR_n, \]
where \( A_n, B_n, C_n, D_n \) are functions of \( x \) defined by

\[
A_n = \frac{2n + \alpha + \beta}{2n(n + \beta)} (1 - x^2), \quad B_n = \frac{2n + \alpha + \beta}{2(n + \beta)} \left( x + \frac{\beta - \alpha}{2n + \alpha + \beta} \right),
\]

\[
C_n = \frac{2n + \alpha + \beta + 2}{2(n + \alpha + 1)(n + \alpha + \beta + 1)} (x^2 - 1),
\]

\[
D_n = \frac{2n + \alpha + \beta + 2}{2(n + \alpha + 1)} \left( x + \frac{\alpha - \beta}{2n + \alpha + \beta + 2} \right).
\]

Using

\[
\left( \frac{R'_n}{R_n} \right)' = \left( R'_n R_n^2 - R_n^{2n} \right) R_n^2
\]

and the differential equation [8, p. 60] satisfied by \( R_n \) we obtain

\[
(1 - x^2)R_n^2 = [x - \beta + (\alpha + \beta + 2)x]R_n R'_n
\]

\[- n(n + \alpha + \beta + 1)R_n^2 - (1 - x^2)(R_n R_n') R_n^2.
\]

Hence

\[
\Delta_n = R_n^2 - (A_n R'_n + B_n R_n)(C_n R'_n + D_n R_n)
\]

\[
= [1 - B_n D_n + n(n + \alpha + \beta + 1)(1 - x^2)^{-1} A_n C_n] R_n^2
\]

\[
+ \left[ -A_n A'_n - B_n C_n + (1 - x^2)^{-1} (\beta - \alpha - (\alpha + \beta + 2)x) A_n C_n \right] R_n R'_n
\]

\[
+ \left( \frac{\beta - \alpha}{2(n + \alpha + 1)(n + \beta + 1)} \right) R_n^2
\]

\[
\times \frac{A_n C_n R_n^2}{1 - x^2}.
\]

Now recall [8, §3.3] that all zeros of \( R_n(x; \alpha, \beta) \) are real and simple and are located in the open interval \((-1, 1)\). Thus

\[
\frac{R'_n}{R_n} = \sum_{k=1}^{n} (x - x_k)^{-1}, \quad \left( \frac{R_n}{R'_n} \right)' = - \sum_{k=1}^{n} (x - x_k)^{-2},
\]

where \( x_1, x_2, \ldots, x_n \) are the zeros of \( R_n \), and so

\[
\Delta_n = \frac{(\beta - \alpha)(1 - x)}{2(n + \alpha + 1)(n + \beta)} R_n^2
\]

\[
- \frac{(1 - x^2)^{-1} A_n C_n R_n^2}{(2n + \alpha + \beta)(2n + \alpha + \beta + 2)} \sum_{k=1}^{n} E(k, n; \alpha, \beta)
\]

\[
\times (x - x_k)^2.
\]
with
\[ E(k, n; \alpha, \beta) = (2n + \alpha + \beta)(2n + \alpha + \beta + 2)(1 - xx_k) + (\alpha^2 - \beta^2)(x - x_k). \]

To complete the proof it suffices to observe that the expression
\[-A_n C_n E(k, n; \alpha, \beta) \text{ in (4) is (strictly) positive since } -A_n C_n > 0 \text{ and} \]
\[ E(k, n; \alpha, \beta) = 2[2(n - 1)(n + \alpha + \beta + 2) + 2(\alpha + 1) \]
\[ + (\beta + 1)(\alpha + \beta + 2)](1 - xx_k) \]
\[ + (\alpha^2 - \beta^2)(1 + x)(1 - x_k) \]
\[ = 2[2(n - 1)(n + \alpha + \beta + 2) + 2(\beta + 1) \]
\[ + (\alpha + 1)(\alpha + \beta + 2)](1 - xx_k) \]
\[ + (\beta^2 - \alpha^2)(1 - x)(1 + x_k) > 0 \]
for \(-1 < x, x_k < 1\).

REFERENCES


DEPARTMENT OF MATHEMATICS, NORTHWESTERN UNIVERSITY, EVANSTON, ILLINOIS 60201