THE CLOSED IDEALS OF SOME DIRICHLET AND HYPO-DIRICHLET ALGEBRAS

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Abstract. We characterize the closed ideals of (i) the Dirichlet algebras discovered by A. Browder and J. Wermer and (ii) the hypo-Dirichlet algebras discovered recently by A. G. Brandstein. Our results show that the class of closed ideals of each of these algebras is surprisingly restricted.

We consider algebras of functions defined by means of singular maps. A homeomorphism \( \varphi \) of one circle onto another of radius \( r \) is called singular if \( \varphi \) maps a Borel set of Lebesgue measure zero onto a set of measure \( 2\pi r \). If \( K \) is a compact subset of \( \mathbb{C} \) with boundary \( \partial K \), we denote by \( A_K \) the subalgebra of \( C(\partial K) \) of all functions which admit continuous extensions to \( K \), analytic on the interior of \( K \).

Let \( \Delta = \{ z \in \mathbb{C} : |z| \leq 1 \} \), the closed unit disc, \( T = \partial \Delta \), and \( A_0 = A_\Delta \). Let \( q \) be a singular homeomorphism of \( T \) onto itself. Define \( A_0(q) = \{ f \in A_0 : f \circ q \in A_0 \} \). Then \( A \) is a Dirichlet algebra (i.e. \( \text{Cl}[\text{Re } A_0(q)] = C_R(T) \) \([3]\), and a maximal closed subalgebra of \( A_0 \) \([1]\).

**Theorem 1.** \( I \) is a closed ideal of \( A_0(q) \) if and only if there exist closed ideals \( I_1 \) and \( I_2 \) of \( A_0 \) such that \( I = \{ f \in I_1 : f \circ q \in I_2 \} \).

Suppose now that \( q \circ q = \text{identity} \). Let \( T/q \) denote the quotient space of \( T \) induced by \( q \). Set \( A_q = \{ f \in A_0 : f \circ q = f \} \). The subalgebra of \( C(T/q) \) induced by \( A_q \) is a Dirichlet algebra on \( T/q \), as well as a maximal closed subalgebra of \( C(T/q) \) \([3]\).

**Theorem 2.** \( I \) is a closed ideal of \( A_q \) if and only if there exists a closed ideal \( J \) of \( A_0 \) such that \( I = A_q \cap J \).

Let \( \Gamma \) be the annulus \( \{ z \in \mathbb{C} : 1 \leq |z| \leq 2 \} \), let \( T' = \{ z \in \mathbb{C} : |z| = 2 \} \), and let \( B_0 = A_\Gamma \). Let \( p \) be a singular homeomorphism of \( T \) onto \( T' \) which is orientation preserving. Define \( B_p = \{ f \in B_0 : f(z) = f(p(z)) \} \), all \( z \) in \( T \). The restriction of \( B_p \) to \( T \) is a hypo-Dirichlet algebra (i.e. there exist \( f_1, \ldots, f_n \) in \( B_p \) whose analytic extensions to \( \Gamma \) are never zero such that the real vector space spanned by \( \text{Re } B_p \) and \( \log |f_1|, \ldots, \log |f_n| \) is dense in \( C_R(T) \) \([2]\).

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Theorem 3. $I$ is a closed ideal of $B_\alpha$ if and only if there exists a closed ideal $J$ of $B_\alpha$ such that $I = B_\alpha \cap J$.

We identify the dual space $C(X)^*$ of $C(X)$, $X$ compact, with the space of all complex, regular, Borel measures on $X$. If $A$ is a subspace of $C(X)$, $A^\perp$ is the space of all $\mu \in C(X)^*$ such that $\int f d\mu = 0$ for each $f \in A$.

If $A$ is a closed subalgebra of $C(X)$ and $I$ is a closed ideal of a subalgebra of $A$, we let $I[A]$ denote the closed ideal of $A$ generated by the functions in $I$.

Lemma 1. Let $A$ and $B$ be closed subalgebras containing $1$ of $C(X)$. Suppose there exists a Borel subset $E$ of $X$ such that $|E| = |X - E| = 0$ for every $\mu \in A^\perp$, $\nu \in B^\perp$. If $I_1$ and $I_2$ are closed ideals of $A$ and $B$ respectively, then $I_1 \cap I_2$ is a closed ideal of $A \cap B$, and all the closed ideals of $A \cap B$ arise in this way.

Proof. Let $I$ be a closed ideal of $A \cap B$. We shall show that $I = I[A] \cap I[B]$. Clearly $I \subseteq I[A] \cap I[B]$. To prove the reverse containment we shall prove $I \subseteq (I[A] \cap I[B])\perp$. Thus, let $\eta \in I^\perp$ and $f \in I(A) \cap I[B]$. We have to show $\int f d\eta = 0$.

Let $f \in I[A] \cap I[B]$. We shall show that $f = I[I] \cap I[B]$. Clearly $I \subseteq I[A] \cap I[B]$. To prove the reverse containment we shall prove $I \subseteq (I[A] \cap I[B])\perp$. Thus, let $\eta \in I^\perp$ and $f \in I(A) \cap I[B]$. We have to show $\int f d\eta = 0$.

Lemma 2. Assume $G$ is finite. Then if $J$ is a closed ideal of $A$, $J_G$ is a closed ideal of $A_G$ and every closed ideal of $A_G$ arises in this way.
PROOF. Let $I$ be a closed ideal of $A_G$. We will prove that

$$I = [\text{Cl}(A \cdot I)]_G.$$  

We show first

$$[\text{Cl}(A \cdot I)]_G = \text{Cl}([A \cdot I])_G.$$  

For suppose that $f \in [\text{Cl}(A \cdot I)]_G$. Then there exists a sequence $\{f_n\}$, $f_n \in A \cdot I$, which converges uniformly to $f$. Let $N$ be equal to the order of $G$. Then, if $F_n = N^{-1} \sum_{g \in G} f_n \circ g$, $F_n \in (A \cdot I)_G$ and $F_n \to N^{-1} \sum_{g \in G} f \circ g = f$. Hence $f \in \text{Cl}([A \cdot I])_G$, proving (2).

Since $I \subseteq [\text{Cl}(A \cdot I)]_G$, in order to prove (1) it suffices to prove that $I \subseteq [\text{Cl}(A \cdot I)]_G$. Thus if $\eta \in I$, we need only show (by (2)), that $\eta$ annihilates $(A \cdot I)_G$. Let $h \in (A \cdot I)_G$, where $h = \sum_{i=1}^p a_r \cdot i_r$, $a_r \in A$, $i_r \in I$. Since $i_r \in A_G$ we have

$$\int h \, d\eta = \frac{1}{N} \int \left( \sum_{g \in G} h \circ g \right) \, d\eta = \frac{1}{N} \int \sum_{r=1}^p (a_r \circ g) \cdot i_r \, d\eta = 0.$$  

Q.E.D.

Proof of Theorem 2. Let $G'$ be the two element group of homeomorphisms of $T$ generated by $q$ (so that $A_q = (A_0)_{G'}$). If $I$ is a closed ideal of $A_q$, we obtain by means of Lemma 2 and Theorem 1

$$I = (I[A_q])_{G'} = (I[A_q] \cap I[A_q])_{G'} = (I[A_q])_{G'} = (I[A_q])_{G'} \cap (I[A_q])_{G'}.$$  

Q.E.D.

Proof of Theorem 3. Define $p_1 : T \cup T' \to T \cup T'$ by $p_1|T = p$, $p_1|T' = p^{-1}$. Then $p_1 \circ p_1 = \text{identity}$, $T \cup T'/p_1$ may be identified with $T$, and if $G''$ denotes the two element group of homeomorphisms of $T \cup T'$ on itself generated by $p_1$, $B_{p_1} = B_{G''}$. Thus the proof of Theorem 3 is analogous to the proof of Theorem 2. Q.E.D.

BIBLIOGRAPHY


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