ON THE COMPARABILITY OF $A^{1/2}$ AND $A^{*1/2}$

ALAN McINTOSH

Abstract. There exists a regularly accretive operator $A$ in a Hilbert space $H$ such that $A^{1/2}$ and $A^{*1/2}$ have different domains. Consequently, the domain of the closed bilinear form corresponding to $A$ is different from the domain of $A^{1/2}$.

1. Introduction. Let $A$ denote a regularly accretive linear operator in a complex Hilbert space $H$. It was shown by T. Kato in [1] that if $\langle x \rangle < \frac{1}{2}$ then the domains of $A^a$ and $A^{a\pi}$ are the same. Kato also showed that this is not necessarily the case if $\langle x \rangle \geq \frac{1}{2}$. In this paper we construct a regularly accretive operator $A$ for which the domain of $A^{*1/2}$ is different from the domain of $A^{1/2}$. We remark that the domain of the closed bilinear form corresponding to such an operator $A$ is also different from the domain of $A^{1/2}$ (see [2]).

In proving the existence of such an operator $A$, we use the following result:

(I) Let $k$ be a natural number. Then there exist bounded selfadjoint operators $U$ and $V$ in a (finite-dimensional) Hilbert space $H$ such that $U$ is positive definite and $\|UV - VU\| \geq k \|UV + VU\|$.

Examples of such operators were constructed by the author when searching for a counterexample to a different problem. (See Result (III) of [4], together with the first comment added in the proofs of [4].)

T. Kato has made the interesting observation that if $Z = UV$, where $U$ and $V$ are operators satisfying (I), then $Z$ has real spectrum (for $Z$ is similar to $U^{1/2} V U^{1/2}$), but the numerical range of $Z$ extends vertically at least $k$ times further than horizontally.

Throughout this paper the scalar field is assumed to be the field $C$ of complex numbers. All operators are assumed to be linear. We remark that a densely-defined maximal accretive operator is regularly accretive if $|\text{Im}(Au, u)| \leq \kappa |\text{Re}(Au, u)|$ for some $\kappa \geq 0$ and all $u \in D(A)$, the domain of $A$.

Received by the editors May 17, 1971.

AMS 1970 subject classifications. Primary 47B44.

Key words and phrases. Regularly accretive operator, square root of an operator, Hilbert space, closed bilinear form.

1 An example of a maximal accretive (but not regularly accretive) operator $A$ with $D(A^{1/2}) \neq D(A^{*1/2})$ was given by Lions in [5]; namely $A = d/dx$ with $D(A) = H^2(0, \infty)$ in the space $H = L^2(0, \infty)$. Indeed it can be shown that every maximal accretive operator $A$ for which $iA$ is maximal symmetric but not selfadjoint has this property. (See Theorem 4.2 of [6].)
An operator $A$ is called invertible if $A$ is one-one, onto, and has continuous inverse.

2. The result.

**Theorem.** Let $\kappa > 0$. There exists a regularly accretive operator $A$ in a Hilbert space $H$ such that $|\Im(Au, u)| \leq \kappa |\Re(Au, u)|$ for all $u \in D(A)$, and $D(A^{1/2}) \neq D(A^{1/2})$.

**Proof.** We first note the following corollary to result (I) above:

(II) Let $0 < \varepsilon < 1$ and let $1 < \kappa < 2$. There exist bounded selfadjoint operators $S$ and $T$ in a Hilbert space $H$ such that $0 < S \leq 1$, $S$ is invertible, $\|ST - TS\| \leq \varepsilon$ and $\|ST - TS\| = K$.

To prove (II), let $k$ be a natural number such that $2\varepsilon^{-1} \leq k < 3\varepsilon^{-1}$ and choose $U$ and $V$ satisfying the properties mentioned in (I). Now set $S = U^{-1}U$ and $T = K\|U\|\|UV - VU\|^{-1}V$.

We now define, for each natural number $n \geq 2$, a bounded operator $A_n$ in a Hilbert space $H_n$, as follows. Let $K = 2 - n^{-1}$ and choose $\varepsilon \leq \frac{1}{2} \kappa (1 + \kappa)^{-1} n^{-1}$. If $S$, $T$ and $H$ are defined as in (II), let $H_n = H$ and $A_n = (S^{-1} + iT)^2$. We now show that $A_n$ has the following properties:

(i) $\Re(A_nu, u) \geq 0$ for all $u \in H_n$;
(ii) $|\Im(A_nu, u)| \leq \kappa |\Re(A_nu, u)|$ for all $u \in H_n$;
(iii) $\Re(A_n^{1/2}u, u) \geq \|u\|^2$ for all $u \in H_n$;
(iv) there exists an element $v \in H_n$ which does not satisfy the formula

\[(n - 1)^{-1/2} \|A_n^{1/2}v\| \leq \|A_n^{1/2}v\| \leq (n - 1)^{1/2} \|A_n^{1/2}v\|.

In proving these properties, we set $\delta = n^{-1}$. Note that $\delta > 2\varepsilon$. Therefore

\[
\|TS\| \leq \frac{1}{2} \|TS + ST\| + \frac{1}{2} \|TS - ST\|
\]

\[
\leq \frac{1}{2} \varepsilon + 1 - \frac{1}{2} \delta < 1 - \frac{1}{2} \delta.
\]

(i) $\Re(A_nu, u) = ((S^{-2} - T^2)u, u)$

\[
\geq (1 - \|TS\|^2) \|S^{-1}u\|^2 > 0 \quad \text{for all } u \in H_n.
\]

(ii) We must prove that

\[
|(S^{-1}T + TS^{-1})u, u| \leq \kappa ((S^{-2} - T^2)u, u) \quad \text{for all } u \in H_n.
\]

Equivalently, setting $v = S^{-1}u$,

\[
|(TS + ST)v, v| + \kappa \|TSv\|^2 \leq \kappa \|v\|^2 \quad \text{for all } v \in H_n.
\]

This follows from the inequality

\[
\|TS + ST\| + \kappa \|TS\|^2 \leq \varepsilon + \kappa (1 + \frac{1}{2}(\varepsilon - \delta))^2
\]

\[
= \varepsilon + \kappa + \kappa \varepsilon + \kappa \delta + \frac{1}{4} \kappa (\varepsilon - \delta)^2
\]

\[
\leq \kappa + (1 + \kappa) \varepsilon - \kappa \delta + \frac{1}{4} \kappa \delta^2 \quad \text{(by the definition of } \varepsilon).\]
(iii) $A_n^{1/2}$ is the unique accretive operator satisfying $(A_n^{1/2})^2 = A_n$ (see [3, p. 281]), so $A_n^{1/2} = S^{-1} + iT$. Hence
\[ \Re(A_n^{1/2}u, u) = (S^{-1}u, u) \geq ||u||^2 \text{ for all } u \in H_n. \]

(iv) Recall that $\|ST - TS\| = 2 - \delta$. Now $i(ST - TS)$ is selfadjoint, so there is an element $u \in H_n$ satisfying either
\[ (\alpha) \quad |(i(ST - TS)u, u) - (2 - \delta)||u||^2| < \delta||u||^2, \text{ or} \]
\[ (\beta) \quad |(-i(ST - TS)u, u) - (2 - \delta)||u||^2| < \delta||u||^2. \]
First suppose that $u$ satisfies $(\alpha)$. Let $v = Su$.
\[ \therefore \quad |(iT(S^{-1} - S^{-1})T_v, v) - (2 - \delta)||S^{-1}v||^2| < \delta||S^{-1}v||^2. \]
\[ \therefore \quad (2S^{-2} + iT(S^{-1} - S^{-1}))v, v) < 2\delta||S^{-1}v||^2. \]

Now, as was proved in (i), $(T^2v, v) < (S^{-2}v, v)$, so
\[ ((S^{-2} + T^2 - iT(S^{-1} - S^{-1}))v, v) < 2\delta||S^{-1}v||^2 < 2\delta((S^{-2} + T^2)v, v) \]
\[ = \delta((S^{-2} + T^2 + iT(S^{-1} - S^{-1}))v, v) \]
\[ + \delta((S^{-2} + T^2 - iT(S^{-1} - S^{-1}))v, v). \]
\[ \therefore \quad ((S^{-2} + T^2 - iT(S^{-1} - S^{-1}))v, v) \]
\[ < \frac{\delta}{1 - \delta} ((S^{-2} + T^2 + iT(S^{-1} - S^{-1}))v, v). \]
\[ \therefore \quad \|(S^{-1} + iT)v||^2 < (n - 1)^{-1}||(S^{-1} + iT(S^{-1} - S^{-1}))(S^{-1} - iT)v, v). \]
\[ \therefore \quad \|A_n^{*1/2}v|| < (n - 1)^{-1/2} \|A_n^{*1/2}v||. \]

On the other hand, if $u$ satisfies $(\beta)$, then $v = Su$ satisfies
\[ \|A_n^{*1/2}v|| < (n - 1)^{-1/2} \|A_n^{*1/2}v||. \]
So (iv) is proved.

Now define $A$ to be the operator $A = \oplus A_n$ in the Hilbert space $H = \oplus H_n$ (where the direct sum is taken over all natural numbers $n \geq 2$). Then $A$ is densely-defined maximal accretive and satisfies $|\Im(Au, u)| \leq \kappa \Re(Au, u)$ for all $u \in D(A)$. Moreover $A^{1/2}$ and hence $A^{*1/2}$ are invertible, and for every $\gamma > 0$ there exists $v \in D(A^{1/2}) \cap D(A^{*1/2})$ which does not satisfy
\[ \gamma^{-1} \|A^{*1/2}v|| \leq \|A^{1/2}v|| \leq \gamma \|A^{*1/2}v||. \]

By applying the lemma below we conclude that $D(A^{1/2}) \neq D(A^{*1/2}).$

**Lemma.** Let $B$ and $C$ be two closed invertible operators in a Hilbert space $H$ such that $D(B) = D(C)$. Then there exists $\gamma > 0$ such that
\[ \gamma^{-1} \|Bu|| \leq \|Cu|| \leq \gamma \|Bu|| \text{ for all } u \in D(B). \]
3. **A stronger result.** It is natural to ask whether stronger conditions on $A$ would imply that $A^{1/2}$ and $A^{*1/2}$ have the same domain. We will now indicate that the following additional condition is not strong enough:

\[ \inf \{ \theta : \text{the numerical range of } A \text{ is contained in a sector of semiangle } \theta \} = 0. \]

In other words, there exists a regularly accretive operator $A$ which satisfies (C), but for which $D(A^{1/2}) \neq D(A^{*1/2})$.

Define the real-valued function $f$ by $f(y) = y(\log \log y)^{1/3}$ if $y > e$; $=0$ if $y \leq e$. We will show that there exists a regularly accretive operator $A$ with $D(A^{1/2}) \neq D(A^{*1/2})$ which satisfies:

\[ f(\| \text{Im}(A u, u) \|) \leq \text{Re}(A u, u) \quad \text{for all } u \in D(A) \text{ with } \|u\| = 1. \]

Since $f$ is increasing, and $df/dy \to \infty$ as $y \to \infty$, an operator which satisfies (D) also satisfies (C).

The operator $A$ is constructed as before but with an extra condition on $\epsilon$. We note first that the operator $U$ constructed in [4] satisfies $2 \leq U \leq 2^m$, where $m = 2(\delta k + 1)^2$. So the operator $S = ||U||^{-1}U$ satisfies

\[ \|S^{-1}\|^2 \leq 2^{2m-2} < \exp 2m < \exp((6k + 1)^2 + 1) \]

\[ < \exp((18e^{-1} + 1)^2 + 1) < \exp 500e^{-2} \]

(because $k < 3e^{-1}$, and $\epsilon < 1$). Hence (using the monotonicity of $f$), if $\|u\| = 1$,

\[ f(\| \text{Im}(A u, u) \|) = f(\|(S^{-1}T + TS^{-1})u, u\|) \]

\[ \leq f(\|TS + ST\| \|S^{-1}u\|^2) \leq f(\epsilon \|S^{-1}u\|^2) \]

\[ = \begin{cases} \epsilon \|S^{-1}u\|^2(\log \log(\epsilon \|S^{-1}u\|^2))^{1/3}, & \text{if } \epsilon \|S^{-1}u\|^2 > e, \\ 0, & \text{otherwise}; \end{cases} \]

\[ < \begin{cases} \epsilon \|S^{-1}u\|^2(\log \log \|S^{-1}\|^2)^{1/3}, & \text{if } \|S^{-1}\|^2 > e, \\ 0, & \text{otherwise}; \end{cases} \]

\[ < 8e^{1/2} \|S^{-1}u\|^2. \]

On the other hand,

\[ \text{Re}(A u, u) = ((S^{-2} - T^2)u, u) \]

\[ \geq \|S^{-1}u\|^2 - \|TS\|^2 \|S^{-1}u\|^2 \]

\[ > \frac{1}{4} \delta \|S^{-1}u\|^2 \quad (\because \|TS\|^2 < 1 - \frac{1}{4} \delta). \]

Now we may choose $\epsilon$ to satisfy $\epsilon \leq (32)^{-3\delta^2}$, in which case

\[ f(\| \text{Im}(A u, u) \|) < \text{Re}(A u, u) \quad \text{for all } u \in H_n \text{ such that } \|u\| = 1. \]
We conclude that the operator $A = \bigoplus A_n$ (which we have already shown to be regularly accretive and satisfy $D(A^{1/2}) \neq D(A^{*1/2})$) satisfies property (D), and hence (C).

**Remark.** Professor W. Kahan has constructed operators $U$ and $V$ satisfying (I) such that $2^{\leq U \leq 2^m}$ where $m = 2^m$ for some constant $c$. Using these operators, together with slightly more care in the estimates, we can replace the function $f$ in (D) by the function $f(y) = y(\log \log y)^\alpha$ if $y > e$; $=0$ if $y \leq e$, for any $\alpha < 1$. It would be interesting to know what the situation is for functions $f$ of faster growth. In particular, it seems reasonable to conjecture that if $A$ is a maximal accretive operator satisfying $|\Im(Au, u)|^p \leq \kappa \Re(Au, u)$ for all $u \in D(A)$ such that $\|u\| = 1$, where $p > 1$ and $\kappa > 0$, then $D(A^{1/2}) = D(A^{*1/2})$. However this question remains open.

**References**


School of Mathematics and Physics, Macquarie University, North Ryde, N.S.W., Australia 2113