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ON THE COMPARABILITY OF $A^{1/2}$ AND $A^{*1/2}$

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ABSTRACT. There exists a regularly accretive operator A in a Hilbert space H such that $A^{1/2}$ and $A^{*1/2}$ have different domains. Consequently, the domain of the closed bilinear form corresponding to A is different from the domain of $A^{1/2}$.

1. **Introduction.** Let A denote a regularly accretive linear operator in a complex Hilbert space H. It was shown by T. Kato in [1] that if $\alpha<\frac{1}{2}$ then the domains of A^α and $A^{*\alpha}$ are the same. Kato also showed that this is not necessarily the case if $\alpha>\frac{1}{2}$. In this paper we construct a regularly accretive operator A for which the domain of $A^{*1/2}$ is different from the domain of $A^{1/2}$. We remark that the domain of the closed bilinear form corresponding to such an operator A is also different from the domain of $A^{1/2}$ (see [2]).[1]

In proving the existence of such an operator A, we use the following result:

(I) Let k be a natural number. Then there exist bounded selfadjoint operators U and V in a (finite-dimensional) Hilbert space H such that U is positive definite and $\|UV-VU\|\geq k\|UV+VU\|$.

Examples of such operators were constructed by the author when searching for a counterexample to a different problem. (See Result (III) of [4], together with the first comment added in the proofs of [4].)

T. Kato has made the interesting observation that if $Z=UV$, where U and V are operators satisfying (I), then Z has real spectrum (for Z is similar to $U^{1/2}\ V\ U^{1/2}$), but the numerical range of Z extends vertically at least k times further than horizontally.

Throughout this paper the scalar field is assumed to be the field $\boldsymbol{C}$ of complex numbers. All operators are assumed to be linear. We remark that a densely-defined maximal accretive operator is regularly accretive if $|\mathrm{Im}(Au, u)|\leq\kappa\ \mathrm{Re}(Au, u)$ for some $\kappa\geq 0$ and all $u\in D(A)$, the domain of A.

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[1] An example of a maximal accretive (but not regularly accretive) operator A with $D(A^{1/2})\neq D(A^{*1/2})$ was given by Lions in [5]; namely $A=d/dx$ with $D(A)=H_0^1(0, \infty)$ in the space $H=L^2(0, \infty)$. Indeed it can be shown that every maximal accretive operator A for which iA is maximal symmetric but not selfadjoint has this property. (See Theorem 4.2 of [6].)

An operator A is called invertible if A is one-one, onto, and has continuous inverse.

2. The result.

THEOREM. *Let $\kappa>0$. There exists a regularly accretive operator A in a Hilbert space H such that $|\mathrm{Im}(Au, u)|\leqq\kappa\,\mathrm{Re}(Au, u)$ for all $u\in D(A)$, and $D(A^{1/2})\neq D(A^{*1/2})$.*

PROOF. We first note the following corollary to result (I) above:

(II) Let $0<\varepsilon<1$ and let $1<K<2$. There exist bounded selfadjoint operators S and T in a Hilbert space H such that $0<S\leqq 1$, S is invertible, $\|ST+TS\|\leqq\varepsilon$ and $\|ST-TS\|=K$.

To prove (II), let k be a natural number such that $2\varepsilon^{-1}\leqq k<3\varepsilon^{-1}$ and choose U and V satisfying the properties mentioned in (I). Now set $S=\|U\|^{-1}U$ and $T=K\|U\|\,\|UV-VU\|^{-1}V$.

We now define, for each natural number $n\geqq 2$, a bounded operator A_n in a Hilbert space H_n, as follows. Let $K=2-n^{-1}$ and choose $\varepsilon\leqq\frac{1}{2}\kappa(1+\kappa)^{-1}n^{-1}$. If S, T and H are defined as in (II), let $H_n=H$ and $A_n=(S^{-1}+iT)^2$. We now show that A_n has the following properties:

(i) $\mathrm{Re}(A_nu, u)\geqq 0$ for all $u\in H_n$;

(ii) $|\mathrm{Im}(A_nu, u)|\leqq\kappa\,\mathrm{Re}(A_nu, u)$ for all $u\in H_n$;

(iii) $\mathrm{Re}(A_n^{1/2}u, u)\geqq\|u\|^2$ for all $u\in H_n$;

(iv) there exists an element $v\in H_n$ which does not satisfy the formula

$$(n-1)^{-1/2}\,\|A_n^{*1/2}v\|\leqq\|A_n^{1/2}v\|\leqq(n-1)^{1/2}\,\|A_n^{*1/2}v\|.$$

In proving these properties, we set $\delta=n^{-1}$. Note that $\delta>2\varepsilon$. Therefore

$$\begin{aligned}\|TS\| &\leqq \tfrac{1}{2}\,\|TS+ST\|+\tfrac{1}{2}\,\|TS-ST\|\\ &\leqq \tfrac{1}{2}\varepsilon+1-\tfrac{1}{2}\delta<1-\tfrac{1}{4}\delta.\end{aligned}$$

(i)
$$\begin{aligned}\mathrm{Re}(A_nu, u) &= ((S^{-2}-T^2)u, u)\\ &\geqq (1-\|TS\|^2)\,\|S^{-1}u\|^2>0 \quad \text{for all } u\in H_n.\end{aligned}$$

(ii) We must prove that

$$|((S^{-1}T+TS^{-1})u, u)|\leqq\kappa((S^{-2}-T^2)u, u)\quad\text{for all } u\in H_n.$$

Equivalently, setting $v=S^{-1}u$,

$$|((TS+ST)v, v)|+\kappa\,\|TSv\|^2\leqq\kappa\,\|v\|^2\quad\text{for all } v\in H_n.$$

This follows from the inequality

$$\begin{aligned}\|TS+ST\|+\kappa\,\|TS\|^2 &\leqq \varepsilon+\kappa\{1+\tfrac{1}{2}(\varepsilon-\delta)\}^2\\ &= \varepsilon+\kappa+\kappa(\varepsilon-\delta)+\tfrac{1}{4}\kappa(\varepsilon-\delta)^2\\ &\leqq \kappa+(1+\kappa)\varepsilon-\kappa\delta+\tfrac{1}{4}\kappa\delta^2 \qquad (\because\ \varepsilon<\delta)\\ &\leqq \kappa \qquad \text{(by the definition of } \varepsilon).\end{aligned}$$

(iii) $A_n^{1/2}$ is the *unique* accretive operator satisfying $(A_n^{1/2})^2=A_n$ (see [3, p. 281]), so $A_n^{1/2}=S^{-1}+iT$. Hence

$$\operatorname{Re}(A_n^{1/2}u, u)=(S^{-1}u, u) \geqq \|u\|^2 \quad \text{for all } u \in H_n.$$

(iv) Recall that $\|ST-TS\|=2-\delta$. Now $i(ST-TS)$ is selfadjoint, so there is an element $u\in H_n$ satisfying either

(α) $|(i(ST-TS)u, u)-(2-\delta)\|u\|^2|<\delta\|u\|^2$, or

(β) $|(-i(ST-TS)u, u)-(2-\delta)\|u\|^2|<\delta\|u\|^2$.

First suppose that u satisfies (α). Let $v=Su$.

$$\therefore\quad |(i(TS^{-1}-S^{-1}T)v, v)-(2-\delta)\|S^{-1}v\|^2| < \delta\|S^{-1}v\|^2.$$

$$\therefore\quad (\{2S^{-2}-i(TS^{-1}-S^{-1}T)\}v, v) < 2\delta\,\|S^{-1}v\|^2.$$

Now, as was proved in (i), $(T^2v, v)<(S^{-2}v, v)$, so

$$\begin{aligned}(\{S^{-2}+T^2-i(TS^{-1}-S^{-1}T)\}v, v)&\\ &< 2\delta\,\|S^{-1}v\|^2 < 2\delta((S^{-2}+T^2)v, v)\\ &= \delta(\{S^{-2}+T^2+i(TS^{-1}-S^{-1}T)\}v, v)\\ &\quad+\delta(\{S^{-2}+T^2-i(TS^{-1}-S^{-1}T)\}v, v).\end{aligned}$$

$$\begin{aligned}\therefore\quad (\{S^{-2}+T^2-i(TS^{-1}-S^{-1}T)\}v, v)&\\ &< \frac{\delta}{1-\delta}(\{S^{-2}+T^2+i(TS^{-1}-S^{-1}T)\}v, v).\end{aligned}$$

$$\therefore\quad ((S^{-1}-iT)(S^{-1}+iT)v, v) < (n-1)^{-1}((S^{-1}+iT)(S^{-1}-iT)v, v).$$

$$\therefore\quad \|(S^{-1}+iT)v\|^2 < (n-1)^{-1}\,\|(S^{-1}-iT)v\|^2.$$

$$\therefore\quad \|A_n^{1/2}v\| < (n-1)^{-1/2}\,\|A_n^{*1/2}v\|.$$

On the other hand, if u satisfies (β), then $v=Su$ satisfies

$$\|A_n^{*1/2}v\| < (n-1)^{-1/2}\,\|A_n^{1/2}v\|.$$

So (iv) is proved.

Now define A to be the operator $A=\bigoplus A_n$ in the Hilbert space $H=\bigoplus H_n$ (where the direct sum is taken over all natural numbers $n\geqq 2$). Then A is densely-defined maximal accretive and satisfies $|\operatorname{Im}(Au, u)|\leqq\kappa\operatorname{Re}(Au, u)$ for all $u\in D(A)$. Moreover $A^{1/2}$ and hence $A^{*1/2}$ are invertible, and for every $\gamma>0$ there exists $v\in D(A^{1/2})\cap D(A^{*1/2})$ which does not satisfy

$$\gamma^{-1}\,\|A^{*1/2}v\| \leqq \|A^{1/2}v\| \leqq \gamma\,\|A^{*1/2}v\|.$$

By applying the lemma below we conclude that $D(A^{1/2})\neq D(A^{*1/2})$.

LEMMA. *Let B and C be two closed invertible operators in a Hilbert space H such that $D(B)=D(C)$. Then there exists $\gamma>0$ such that*

$$\gamma^{-1}\,\|Bu\| \leqq \|Cu\| \leqq \gamma\,\|Bu\| \quad \textit{for all } u \in D(B).$$

3. **A stronger result.** It is natural to ask whether stronger conditions on A would imply that $A^{1/2}$ and $A^{*1/2}$ have the same domain. We will now indicate that the following additional condition is not strong enough:

$$\text{(C)}\quad \inf\{\theta \mid \text{the numerical range of } A \text{ is contained in a sector of semiangle } \theta\} = 0.$$

In other words, there exists a regularly accretive operator A which satisfies (C), but for which $D(A^{1/2}) \neq D(A^{*1/2})$.

Define the real-valued function f by $f(y) = y(\log\log y)^{1/3}$ if $y > e$; $=0$ if $y \leqq e$. We will show that there exists a regularly accretive operator A with $D(A^{1/2}) \neq D(A^{*1/2})$ which satisfies:

$$\text{(D)}\quad f(|\mathrm{Im}(Au, u)|) \leqq \mathrm{Re}(Au, u) \quad \text{for all } u \in D(A) \text{ with } \|u\| = 1.$$

Since f is increasing, and $df/dy \to \infty$ as $y \to \infty$, an operator which satisfies (D) also satisfies (C).

The operator A is constructed as before but with an extra condition on ε. We note first that the operator U constructed in [4] satisfies $2 \leqq U \leqq 2^m$, where $m = 2^{(6k+1)^2}$. So the operator $S = \|U\|^{-1}U$ satisfies

$$\begin{aligned}\|S^{-1}\|^2 \leqq 2^{2m-2} < \exp 2m &< \exp\exp((6k+1)^2 + 1) \\ &< \exp\exp((18\varepsilon^{-1} + 1)^2 + 1) < \exp\exp 500\varepsilon^{-2}\end{aligned}$$

(because $k < 3\varepsilon^{-1}$, and $\varepsilon < 1$). Hence (using the monotonicity of f), if $\|u\| = 1$,

$$\begin{aligned} f(|\mathrm{Im}(A_n u, u)|) &= f(|((S^{-1}T + TS^{-1})u, u)|) \\ &\leqq f(\|TS + ST\| \, \|S^{-1}u\|^2) \leqq f(\varepsilon \, \|S^{-1}u\|^2) \\ &= \begin{cases} \varepsilon \, \|S^{-1}u\|^2 \{\log\log(\varepsilon \, \|S^{-1}u\|^2)\}^{1/3}, & \text{if } \varepsilon \, \|S^{-1}u\|^2 > e, \\ 0, & \text{otherwise;} \end{cases} \\ &< \begin{cases} \varepsilon \, \|S^{-1}u\|^2 \{\log\log \|S^{-1}\|^2\}^{1/3}, & \text{if } \|S^{-1}\|^2 > e, \\ 0, & \text{otherwise;} \end{cases} \\ &< 8\varepsilon^{1/3} \, \|S^{-1}u\|^2. \end{aligned}$$

On the other hand,

$$\begin{aligned} \mathrm{Re}(A_n u, u) &= ((S^{-2} - T^2)u, u) \\ &\geqq \|S^{-1}u\|^2 - \|TS\|^2 \, \|S^{-1}u\|^2 \\ &> \tfrac{1}{4}\delta \, \|S^{-1}u\|^2 \qquad (\because \|TS\|^2 < 1 - \tfrac{1}{4}\delta). \end{aligned}$$

Now we may choose ε to satisfy $\varepsilon \leqq (32)^{-3}\delta^3$, in which case

$$f(|\mathrm{Im}(A_n u, u)|) < \mathrm{Re}(A_n u, u) \quad \text{for all } u \in H_n \text{ such that } \|u\| = 1.$$

We conclude that the operator $A=\bigoplus A_n$ (which we have already shown to be regularly accretive and satisfy $D(A^{1/2})\neq D(A^{*1/2})$) satisfies property (D), and hence (C).

REMARK. Professor W. Kahan has constructed operators U and V satisfying (I) such that $2\leqq U\leqq 2^m$ where $m=2^{ck}$ for some constant c. Using these operators, together with slightly more care in the estimates, we can replace the function f in (D) by the function $f(y)=y(\log\log y)^\alpha$ if $y > e$; $=0$ if $y\leqq e$, for any $\alpha<1$. It would be interesting to know what the situation is for functions f of faster growth. In particular, it seems reasonable to conjecture that if A is a maximal accretive operator satisfying $|\mathrm{Im}(Au, u)|^p\leqq\kappa\ \mathrm{Re}(Au, u)$ for all $u\in D(A)$ such that $\|u\|=1$, where $p>1$ and $\kappa>0$, then $D(A^{1/2})=D(A^{*1/2})$. However this question remains open.

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