

RIEMANN SURFACES IN COMPLEX PROJECTIVE SPACES

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ABSTRACT. The complex projective line and the complex quadric are the only compact Riemann surfaces in the complex projective plane with constant scalar normal curvature.

1. Introduction. Let M^n be an n -dimensional manifold immersed in an $(n+p)$ -dimensional Riemannian manifold R^{n+p} . The scalar normal curvature K_N of M^n in R^{n+p} is defined as the length of the curvature form of the connection in the normal bundle (for the details; see §2). It seems to the authors that the scalar normal curvature is important for the theory of submanifolds of higher codimensions. In [1], [2], [4], Itoh and the authors studied the scalar normal curvature for submanifolds of real space forms. In the present paper, we shall study the scalar normal curvature for submanifolds of a Kähler manifold. In particular, we have

MAIN THEOREM. *The complex projective line CP^1 and the complex quadric Q^1 are the only compact Riemann surfaces in the complex projective plane with constant scalar normal curvature.*

2. Local formulas. In this section, we shall compute some elementary formulas for later use and define the scalar normal curvature.

Let M be an n -dimensional manifold immersed in an $(n+p)$ -dimensional Riemannian manifold N^{n+p} . We choose a local field of orthonormal frames e_1, \dots, e_{n+p} in N^{n+p} such that, restricted to M , the vectors e_1, \dots, e_n are tangent to M (and, consequently, e_{n+1}, \dots, e_{n+p} are normal to M). We shall make use of the following convention on the ranges of indices:

$$\begin{aligned} 1 \leq A, B, C, \dots \leq n+p; & \quad 1 \leq i, j, k, \dots \leq n; \\ n+1 \leq \alpha, \beta, \gamma, \dots \leq n+p; & \end{aligned}$$

and we shall agree that repeated indices are summed over the respective ranges. With respect to the frame field of N^{n+p} chosen above, let $\omega^1, \dots, \omega^{n+p}$ be the field of dual frames. Then the structure equations of N^{n+p} are

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given by

$$(2.1) \quad d\omega^A = -\sum \omega_B^A \wedge \omega^B, \quad \omega_B^A + \omega_A^B = 0,$$

$$(2.2) \quad d\omega_B^A = -\sum \omega_C^A \wedge \omega_B^C + \Phi_B^A, \quad \Phi_B^A = \frac{1}{2} \sum K_{BCD}^A \omega^C \wedge \omega^D,$$

$$K_{BCD}^A + K_{BDC}^A = 0.$$

We restrict these forms to M . Then $\omega^\alpha = 0$. Since $0 = d\omega^\alpha = -\sum \omega_i^\alpha \wedge \omega^i$, by Cartan's lemma we may write

$$(2.3) \quad \omega_i^\alpha = \sum h_{ij}^\alpha \omega^j, \quad h_{ij}^\alpha = h_{ji}^\alpha.$$

From these formulas, we obtain

$$(2.4) \quad d\omega^i = -\sum \omega_j^i \wedge \omega^j, \quad \omega_j^i + \omega_i^j = 0,$$

$$(2.5) \quad d\omega_j^i = -\sum \omega_k^i \wedge \omega_j^k + \Omega_j^i, \quad \Omega_j^i = \frac{1}{2} \sum R_{jkl}^i \omega^k \wedge \omega^l,$$

$$(2.6) \quad R_{jkl}^i = K_{jkl}^i + \sum_\alpha (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha),$$

$$(2.7) \quad d\omega_\beta^\alpha = -\sum \omega_\gamma^\alpha \wedge \omega_\beta^\gamma + \Omega_\beta^\alpha, \quad \Omega_\beta^\alpha = \frac{1}{2} \sum R_{\beta kl}^\alpha \omega^k \wedge \omega^l,$$

$$(2.8) \quad R_{\beta kl}^\alpha = K_{\beta kl}^\alpha + \sum_i (h_{ik}^\alpha h_{il}^\beta - h_{il}^\alpha h_{ik}^\beta).$$

We call $\sum h_{ij}^\alpha \omega^i \omega^j e_\alpha$ the *second fundamental form* of the immersed manifold M . The mean curvature vector is given by $(1/n) \sum_\alpha (\sum_i h_{ii}^\alpha) e_\alpha$.

We denote by K_N the length of the curvature form of the normal connection, i.e.,

$$(2.9) \quad K_N = \sum S_{\beta ij}^\alpha S_{\beta ij}^\alpha, \quad S_{\beta ij}^\alpha = \sum (h_{ik}^\alpha h_{jk}^\beta - h_{jk}^\alpha h_{ik}^\beta).$$

We call K_N the *scalar normal curvature* of this immersion.

The forms (ω_j^i) define a connection in the tangent bundle $T = T(M)$ and the (ω_β^α) define a connection in the normal bundle $T^\perp = T^\perp(M)$. Consequently, we have covariant differentiation which maps a section of $T^\perp \otimes T^* \otimes \dots \otimes T^*$, (T^* ; k times), into a section of $T^\perp \otimes T^* \otimes \dots \otimes T^* \otimes T^*$, (T^* ; $k+1$ times). The second fundamental form h_{ij}^α is a section of the vector bundle $T^\perp \otimes T^* \otimes T^*$.

We take exterior differentiation of (2.3) and define h_{ijk}^α by

$$(2.10) \quad \sum h_{ijk}^\alpha \omega^k = dh_{ij}^\alpha - \sum h_{il}^\alpha \omega_j^l - \sum h_{lj}^\alpha \omega_i^l + \sum h_{ij}^\beta \omega_\beta^\alpha.$$

Then

$$(2.11) \quad \sum (h_{ijk}^\alpha + \frac{1}{2} K_{ijk}^\alpha) \omega^j \wedge \omega^k = 0.$$

We take exterior differentiation of (2.10) and define h_{ijk}^α by

$$(2.12) \quad \sum h_{ijk}^\alpha \omega^l = dh_{ijk}^\alpha - \sum h_{ijk}^\alpha \omega_i^l - \sum h_{ijk}^\alpha \omega_j^l - \sum h_{ijk}^\alpha \omega_k^l + \sum h_{ijk}^\beta \omega_\beta^\alpha.$$

The Laplacian Δh_{ij}^α of the second fundamental form h_{ij}^α is defined by

$$(2.13) \quad \Delta h_{ij}^\alpha = \sum_k h_{ijk}^\alpha.$$

By a straightforward, simple calculation we have [3]

$$(2.14) \quad \begin{aligned} \Delta h_{ij}^\alpha = & \sum_k h_{kij}^\alpha + \sum_{\beta, k} (-K_{ij\beta}^\alpha h_{kk}^\beta + 2K_{\beta ki}^\alpha h_{jk}^\beta - K_{k\beta k}^\alpha h_{ij}^\beta + 2K_{\beta k j}^\alpha h_{ki}^\beta) \\ & + \sum_{m, k} (K_{kik}^m h_{mj}^\alpha + K_{kjk}^m h_{mi}^\alpha + 2K_{ijk}^m h_{mk}^\alpha) \\ & + \sum_{\beta, m, k} (h_{mi}^\alpha h_{mj}^\beta h_{kk}^\beta + 2h_{km}^\alpha h_{ki}^\beta h_{mj}^\beta - h_{km}^\alpha h_{km}^\beta h_{ij}^\beta \\ & \qquad \qquad \qquad - h_{mi}^\alpha h_{mk}^\beta h_{kj}^\beta - h_{mj}^\alpha h_{ki}^\beta h_{mk}^\beta). \end{aligned}$$

3. Proof of Main Theorem. Let CP^m be the complex projective space of complex dimension $m=1+p/2$ with the Fubini-Study metric. Let M be a compact Riemann surface holomorphically immersed in CP^m , and J be the complex structure of CP^m . Then M is a minimal submanifold in CP^m and the curvature tensor of CP^m is given by

$$(3.1) \quad K_{BCD}^A = \frac{1}{4}(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC} + J_{AC}J_{BD} - J_{AD}J_{BC} + 2J_{AB}J_{CD}).$$

By a straightforward, simple calculation, formula (2.23) of [3] reduces to

$$(3.2) \quad \begin{aligned} \sum h_{ij}^\alpha \Delta h_{ij}^\alpha = & -\sum (h_{ik}^\alpha h_{kj}^\beta - h_{ik}^\beta h_{kj}^\alpha)(h_{il}^\alpha h_{lj}^\beta - h_{il}^\beta h_{lj}^\alpha) \\ & - \sum h_{ij}^\alpha h_{ki}^\alpha h_{ij}^\beta h_{kl}^\beta + \frac{3}{2} \sum h_{ij}^\alpha h_{ij}^\alpha. \end{aligned}$$

Similarly, formula (2.22) of [3] gives

$$(3.3) \quad \begin{aligned} \frac{1}{2}\Delta K_N = & \sum (S_{\beta ijk}^\alpha)^2 + 2 \sum S_{\beta ij}^\alpha (h_{ikt}^\alpha h_{jkl}^\beta - h_{ikt}^\beta h_{jkl}^\alpha) + 4 \sum S_{\beta ij}^\alpha (\Delta h_{ik}^\alpha) h_{jk}^\beta \\ = & \sum (S_{\beta ijk}^\alpha)^2 + 2 \sum S_{\beta ij}^\alpha (h_{ikt}^\alpha h_{jkl}^\beta - h_{ikt}^\beta h_{jkl}^\alpha) \\ & + (3 - \sum h_{ij}^\alpha h_{ij}^\alpha) K_N + 8 \sum S_{\beta ij}^\alpha h_{im}^\alpha h_{mk}^\beta h_{il}^\beta h_{jk}^\beta \\ & - 4 \sum S_{\beta ij}^\alpha [\sum h_{im}^\alpha h_{lm}^\beta h_{ik}^\beta h_{jk}^\beta + \sum h_{mk}^\alpha h_{il}^\beta h_{ml}^\beta h_{jk}^\beta]. \end{aligned}$$

For a matrix $A=(a_{ij})$ we denote by $N(A)$ the square of the norm of A , i.e.,

$$N(A) = \text{trace } A \cdot {}^t A = \sum (a_{ij})^2.$$

For each α , let H_α denote the symmetric matrix (h_{ij}^α) and set

$$(3.4) \quad S_\alpha = N(H_\alpha), \quad S = \sum S_\alpha.$$

Then (3.1) reduces to

$$(3.5) \quad \sum h_{ij}^\alpha \Delta h_{ij}^\alpha = -\sum N(H_\alpha H_\beta - H_\beta H_\alpha) - \sum S_\alpha^2 + \frac{3}{2}S.$$

From now on, we assume that the scalar normal curvature K_N of M in CP^m is constant. We now consider the cases $K_N=0$ and $K_N \neq 0$ separately.

Case $K_N \neq 0$. We define a linear mapping φ from the normal space T_p^\perp at p into the space of all symmetric matrices of order 2 by

$$\varphi\left(\sum v_\alpha e_\alpha\right) = \sum v_\alpha H_\alpha.$$

Set $O_p = \varphi^{-1}(0)$. Then by the assumption $K_N \neq 0$ and $\sum h_{ii}^\alpha = 0$, we see that $\dim O_p = 2m - 4$. Let N'_p be the subspace of T_p^\perp given by

$$(3.6) \quad T_p^\perp = N'_p \oplus O_p, \quad N'_p \perp O_p.$$

Then $\dim N'_p = 2$ and this decomposition is well defined and smooth. In the following, we always choose $e_3, e_4 \in N'_p$. Then we have

$$(3.7) \quad h_{ij}^\gamma = 0 \quad \text{for } \gamma > 4.$$

Hence, (3.3) reduces to

$$(3.8) \quad \begin{aligned} \frac{1}{2}\Delta K_N &= 2 \sum (S_{\beta ijk}^\alpha)^2 + 4 \sum S_{4ij}^3 (h_{ikl}^3 h_{jkl}^4 - h_{ikl}^4 h_{jkl}^3) \\ &+ (3 - S)K_N - 2 \text{trace} (RH_3 - H_3R)^2 \\ &- 2 \text{trace} (RH_4 - H_4R)^2, \end{aligned}$$

where $R = (S_{4ij}^3)$. We choose e_1, e_2 in the principal directions of e_3 and $e_2 = J e_1$. Then we have

$$(3.9) \quad H_3 = \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix}, \quad H_4 = \begin{bmatrix} d & e \\ e & -d \end{bmatrix}, \quad a \neq 0, d^2 + e^2 \neq 0,$$

$$(3.10) \quad R = (S_{4ij}^3) = \begin{bmatrix} 0 & 2ae \\ -2ae & 0 \end{bmatrix},$$

$$(3.11) \quad S = 2a^2 + 2d^2 + 2e^2,$$

$$(3.12) \quad K_N = 16a^2e^2 = \text{constant} > 0,$$

$$(3.13) \quad \text{trace} (RH_3 - H_3R)^2 = 32a^4e^2 = 2a^2K_N,$$

$$(3.14) \quad \text{trace} (RH_4 - H_4R)^2 = 2(d^2 + e^2)K_N.$$

Hence, (3.8) reduces to

$$(3.15) \quad \frac{1}{2}\Delta K_N = 4 \sum S_{4ij}^3 (h_{ikl}^3 h_{jkl}^4 - h_{ikl}^4 h_{jkl}^3) + 3(1 - S)K_N + \sum (S_{\beta ijk}^\alpha)^2.$$

If $m = 2$, then from (3.12) and (3.15) we obtain

$$(3.16) \quad K_N(1 - S) = \frac{3}{2} \sum_l (a_{,l}e)^2 \geq 0,$$

where $a_{,l}$ denotes the covariant derivative of a . By Lemma 1 of [3], (3.5) and (3.16) we obtain

$$(3.17) \quad \begin{aligned} \sum h_{ij}^\alpha \Delta h_{ij}^\alpha &\geq -S_3^2 - S_4^2 + \frac{3}{2}S - 4S_3S_4 \\ &= \frac{3}{2}(1 - S)S + \frac{1}{2}(S_3 - S_4)^2 \\ &\geq \frac{1}{2}(S_3 - S_4)^2 \geq 0. \end{aligned}$$

Hence we obtain

$$(3.18) \quad \frac{1}{2}\Delta S = \sum (h_{ijk}^\alpha)^2 + \sum h_{ij}^\alpha \Delta h_{ij}^\alpha \geq 0.$$

By Hopf's lemma, we see that S is constant and

$$(3.19) \quad h_{ijk}^\alpha = 0,$$

$$(3.20) \quad \sum h_{ij}^\alpha \Delta h_{ij}^\alpha = 0.$$

Formula (3.20) implies

$$(3.21) \quad N(H_3H_4 - H_4H_3)^2 = 2N(H_3)N(H_4),$$

$$(3.22) \quad S_3 = S_4 = \frac{1}{2}.$$

Therefore, by Lemma 1 of [3] we see that H_3 and H_4 are scalar multiples of

$$\tilde{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

respectively. Hence, we may conclude that $d=0$ and

$$(3.23) \quad H_3 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}, \quad H_4 = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}.$$

On the other hand, since $e_2 = Je_1$, we can easily show that the second fundamental form in the direction of Je_3 is given by

$$\begin{bmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix}.$$

Since $h_{ijk}^\alpha = 0$, we have

$$(3.24) \quad dh_{ij}^\alpha = \sum h_{ik}^\alpha \omega_j^k + \sum h_{kj}^\alpha \omega_i^k - \sum h_{ij}^\beta \omega_\beta^\alpha.$$

Using (3.23) and (3.24) we can prove that

$$(3.25) \quad \omega_3^4 = 2\omega_1^2.$$

Therefore, we may conclude that M is a complex quadric in CP^2 .

Case $K_N=0$. This case is contained in the following:

PROPOSITION. *Let M be a complex submanifold of a Kähler manifold \tilde{M} with complex structure J . Then the scalar normal curvature $K_N=0$ at $p \in M$ if and only if the second fundamental form $h=0$ at p .*

PROOF. Let A be the second fundamental form in the direction e . Then if $K_N=0$ at p , we have

$$N(AJA - JAA) = 0.$$

This implies that

$$(3.26) \quad \text{trace}(AJA^t(AJA)) = \text{trace} A^4 = 0.$$

By (3.26) and $A=^tA$, we obtain $A=0$ at p . Consequently, $h=0$ at p . The converse of this is trivial. This proves the proposition. Hence the Main Theorem is proved completely.

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