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RIEMANN SURFACES IN COMPLEX PROJECTIVE SPACES

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ABSTRACT. The complex projective line and the complex quadric are the only compact Riemann surfaces in the complex projective plane with constant scalar normal curvature.

1. **Introduction.** Let M^n be an n-dimensional manifold immersed in an $(n+p)$-dimensional Riemannian manifold R^{n+p}. The scalar normal curvature K_N of M^n in R^{n+p} is defined as the length of the curvature form of the connection in the normal bundle (for the details; see §2). It seems to the authors that the scalar normal curvature is important for the theory of submanifolds of higher codimensions. In [1], [2], [4], Itoh and the authors studied the scalar normal curvature for submanifolds of real space forms. In the present paper, we shall study the scalar normal curvature for submanifolds of a Kähler manifold. In particular, we have

MAIN THEOREM. *The complex projective line CP^1 and the complex quadric Q^1 are the only compact Riemann surfaces in the complex projective plane with constant scalar normal curvature.*

2. **Local formulas.** In this section, we shall compute some elementary formulas for later use and define the scalar normal curvature.

Let M be an n-dimensional manifold immersed in an $(n+p)$-dimensional Riemannian manifold N^{n+p}. We choose a local field of orthonormal frames $e_1, \cdots, e_{n+p}$ in N^{n+p} such that, restricted to M, the vectors $e_1, \cdots, e_n$ are tangent to M (and, consequently, $e_{n+1}, \cdots, e_{n+p}$ are normal to M). We shall make use of the following convention on the ranges of indices:

$$1 \leqq A, B, C, \cdots \leqq n+p; \qquad 1 \leqq i, j, k, \cdots \leqq n;$$
$$n+1 \leqq \alpha, \beta, \gamma, \cdots \leqq n+p;$$

and we shall agree that repeated indices are summed over the respective ranges. With respect to the frame field of N^{n+p} chosen above, let $\omega^1, \cdots, \omega^{n+p}$ be the field of dual frames. Then the structure equations of N^{n+p} are

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given by

$$\text{(2.1)} \qquad d\omega^A = -\sum \omega_B^A \wedge \omega^B, \qquad \omega_B^A + \omega_A^B = 0,$$

$$\text{(2.2)} \qquad \begin{gathered} d\omega_B^A = -\sum \omega_C^A \wedge \omega_B^C + \Phi_B^A, \qquad \Phi_B^A = \tfrac{1}{2}\sum K_{BCD}^A \omega^C \wedge \omega^D, \\ K_{BCD}^A + K_{BDC}^A = 0. \end{gathered}$$

We restrict these forms to M. Then $\omega^\alpha=0$. Since $0=d\omega^\alpha=-\sum \omega_i^\alpha\wedge\omega^i$, by Cartan's lemma we may write

$$\text{(2.3)} \qquad \omega_i^\alpha = \sum h_{ij}^\alpha \omega^j, \qquad h_{ij}^\alpha = h_{ji}^\alpha.$$

From these formulas, we obtain

$$\text{(2.4)} \qquad d\omega^i = -\sum \omega_j^i \wedge \omega^j, \qquad \omega_j^i + \omega_i^j = 0,$$

$$\text{(2.5)} \qquad d\omega_j^i = -\sum \omega_k^i \wedge \omega_j^k + \Omega_j^i, \qquad \Omega_j^i = \tfrac{1}{2}\sum R_{jkl}^i \omega^k \wedge \omega^l,$$

$$\text{(2.6)} \qquad R_{jkl}^i = K_{jkl}^i + \sum_\alpha (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha),$$

$$\text{(2.7)} \qquad d\omega_\beta^\alpha = -\sum \omega_\gamma^\alpha \wedge \omega_\beta^\gamma + \Omega_\beta^\alpha, \qquad \Omega_\beta^\alpha = \tfrac{1}{2}\sum R_{\beta kl}^\alpha \omega^k \wedge \omega^l,$$

$$\text{(2.8)} \qquad R_{\beta kl}^\alpha = K_{\beta kl}^\alpha + \sum_i (h_{ik}^\alpha h_{il}^\beta - h_{il}^\alpha h_{ik}^\beta).$$

We call $\sum h_{ij}^\alpha \omega^i \omega^j e_\alpha$ the *second fundamental form* of the immersed manifold M. The mean curvature vector is given by $(1/n)\sum_\alpha (\sum_i h_{ii}^\alpha) e_\alpha$.

We denote by K_N the length of the curvature form of the normal connection, i.e.,

$$\text{(2.9)} \qquad K_N = \sum S_{\beta ij}^\alpha S_{\beta ij}^\alpha, \qquad S_{\beta ij}^\alpha = \sum (h_{ik}^\alpha h_{jk}^\beta - h_{jk}^\alpha h_{ik}^\beta).$$

We call K_N the *scalar normal curvature* of this immersion.

The forms (ω_j^i) define a connection in the tangent bundle $T=T(M)$ and the (ω_β^α) define a connection in the normal bundle $T^\perp=T^\perp(M)$. Consequently, we have covariant differentiation which maps a section of $T^\perp\otimes T^*\otimes\cdots\otimes T^*$, ($T^*$; k times), into a section of $T^\perp\otimes T^*\otimes\cdots\otimes T^*\otimes T^*$, ($T^*$; $k+1$ times). The second fundamental form h_{ij}^α is a section of the vector bundle $T^\perp\otimes T^*\otimes T^*$.

We take exterior differentiation of (2.3) and define h_{ijk}^α by

$$\text{(2.10)} \qquad \sum h_{ijk}^\alpha \omega^k = dh_{ij}^\alpha - \sum h_{il}^\alpha \omega_j^l - \sum h_{lj}^\alpha \omega_i^l + \sum h_{ij}^\beta \omega_\beta^\alpha.$$

Then

$$\text{(2.11)} \qquad \sum (h_{ijk}^\alpha + \tfrac{1}{2}K_{ijk}^\alpha)\omega^j \wedge \omega^k = 0.$$

We take exterior differentiation of (2.10) and define h^α_{ijkl} by

$$\sum h^\alpha_{ijkl}\omega^l = dh^\alpha_{ijk} - \sum h^\alpha_{ljk}\omega^l_i - \sum h^\alpha_{ilk}\omega^l_j - \sum h^\alpha_{ijl}\omega^l_k + \sum h^\beta_{ijk}\omega^\alpha_\beta. \tag{2.12}$$

The Laplacian Δh^α_{ij} of the second fundamental form h^α_{ij} is defined by

$$\Delta h^\alpha_{ij} = \sum_k h^\alpha_{ijkk}. \tag{2.13}$$

By a straightforward, simple calculation we have [3]

$$\begin{aligned}\Delta h^\alpha_{ij} = {} & \sum_k h^\alpha_{kkij} + \sum_{\beta,k} (-K^\alpha_{ij\beta}h^\beta_{kk} + 2K^\alpha_{\beta ki}h^\beta_{jk} - K^\alpha_{k\beta k}h^\beta_{ij} + 2K^\alpha_{\beta kj}h^\beta_{ki}) \\ & + \sum_{m,k} (K^m_{kik}h^\alpha_{mj} + K^m_{kjk}h^\alpha_{mi} + 2K^m_{ijk}h^\alpha_{mk}) \\ & + \sum_{\beta,m,k} (h^\alpha_{mi}h^\beta_{mj}h^\beta_{kk} + 2h^\alpha_{km}h^\beta_{ki}h^\beta_{mj} - h^\alpha_{km}h^\beta_{km}h^\beta_{ij} \\ & \qquad - h^\alpha_{mi}h^\beta_{mk}h^\beta_{kj} - h^\alpha_{mj}h^\beta_{ki}h^\beta_{mk}).\end{aligned} \tag{2.14}$$

3. **Proof of Main Theorem.** Let CP^m be the complex projective space of complex dimension $m=1+p/2$ with the Fubini-Study metric. Let M be a compact Riemann surface holomorphically immersed in CP^m, and J be the complex structure of CP^m. Then M is a minimal submanifold in CP^m and the curvature tensor of CP^m is given by

$$K^A_{BCD} = \tfrac{1}{4}(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC} + J_{AC}J_{BD} - J_{AD}J_{BC} + 2J_{AB}J_{CD}). \tag{3.1}$$

By a straightforward, simple calculation, formula (2.23) of [3] reduces to

$$\begin{aligned}\sum h^\alpha_{ij}\Delta h^\alpha_{ij} = {} & -\sum (h^\alpha_{ik}h^\beta_{kj} - h^\beta_{ik}h^\alpha_{kj})(h^\alpha_{il}h^\beta_{lj} - h^\beta_{il}h^\alpha_{lj}) \\ & - \sum h^\alpha_{ij}h^\alpha_{kl}h^\beta_{ij}h^\beta_{kl} + \tfrac{3}{2}\sum h^\alpha_{ij}h^\alpha_{ij}.\end{aligned} \tag{3.2}$$

Similarly, formula (2.22) of [3] gives

$$\begin{aligned}\tfrac{1}{2}\Delta K_N & = \sum (S^\alpha_{\beta ijk})^2 + 2\sum S^\alpha_{\beta ij}(h^\alpha_{ikl}h^\beta_{jkl} - h^\beta_{ikl}h^\alpha_{jkl}) + 4\sum S^\alpha_{\beta ij}(\Delta h^\alpha_{ik})h^\beta_{jk} \\ & = \sum (S^\alpha_{\beta ijk})^2 + 2\sum S^\alpha_{\beta ij}(h^\alpha_{ikl}h^\beta_{jkl} - h^\beta_{ikl}h^\alpha_{jkl}) \\ & \quad + \left(3 - \sum h^\alpha_{ij}h^\alpha_{ij}\right)K_N + 8\sum S^\alpha_{\beta ij}h^\alpha_{lm}h^\gamma_{mk}h^\gamma_{il}h^\beta_{jk} \\ & \quad - 4\sum S^\alpha_{\beta ij}\left[\sum h^\alpha_{lm}h^\gamma_{lm}h^\gamma_{ik}h^\beta_{jk} + \sum h^\alpha_{mk}h^\gamma_{li}h^\gamma_{ml}h^\beta_{jk}\right].\end{aligned} \tag{3.3}$$

For a matrix $A=(a_{ij})$ we denote by $N(A)$ the square of the norm of A, i.e.,

$$N(A) = \text{trace } A \cdot {}^tA = \sum (a_{ij})^2.$$

For each α, let H_α denote the symmetric matrix (h^α_{ij}) and set

$$S_\alpha = N(H_\alpha), \qquad S = \sum S_\alpha. \tag{3.4}$$

Then (3.1) reduces to

$$\sum h_{ij}^{\alpha}\Delta h_{ij}^{\alpha} = -\sum N(H_\alpha H_\beta - H_\beta H_\alpha) - \sum S_\alpha^2 + \tfrac{3}{2}S. \tag{3.5}$$

From now on, we assume that the scalar normal curvature K_N of M in CP^m is constant. We now consider the cases $K_N=0$ and $K_N\neq 0$ separately.

Case $K_N\neq 0$. We define a linear mapping φ from the normal space $T_p^\perp$ at p into the space of all symmetric matrices of order 2 by

$$\varphi\left(\sum \nu_\alpha e_\alpha\right) = \sum \nu_\alpha H_\alpha.$$

Set $O_p=\varphi^{-1}(0)$. Then by the assumption $K_N\neq 0$ and $\sum h_{ii}^{\alpha}=0$, we see that $\dim O_p=2m-4$. Let N_p' be the subspace of $T_p^\perp$ given by

$$T_p^\perp = N_p' \oplus O_p, \qquad N_p' \perp O_p. \tag{3.6}$$

Then $\dim N_p'=2$ and this decomposition is well defined and smooth. In the following, we always choose e_3, $e_4 \in N_p'$. Then we have

$$h_{ij}^{\gamma} = 0 \quad \text{for } \gamma > 4. \tag{3.7}$$

Hence, (3.3) reduces to

$$\begin{aligned}\tfrac{1}{2}\Delta K_N = {} & 2\sum (S_{\beta ijk}^{\alpha})^2 + 4\sum S_{4ij}^{3}(h_{ikl}^{3}h_{jkl}^{4} - h_{ikl}^{4}h_{jkl}^{3}) \\ & + (3-S)K_N - 2\operatorname{trace}(RH_3 - H_3R)^2 \\ & - 2\operatorname{trace}(RH_4 - H_4R)^2,\end{aligned} \tag{3.8}$$

where $R=(S_{4ij}^3)$. We choose e_1, e_2 in the principal directions of e_3 and $e_2=Je_1$. Then we have

$$H_3 = \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix}, \qquad H_4 = \begin{bmatrix} d & e \\ e & -d \end{bmatrix}, \qquad a\neq 0,\ d^2+e^2\neq 0, \tag{3.9}$$

$$R = (S_{4ij}^3) = \begin{bmatrix} 0 & 2ae \\ -2ae & 0 \end{bmatrix}, \tag{3.10}$$

$$S = 2a^2 + 2d^2 + 2e^2, \tag{3.11}$$

$$K_N = 16a^2e^2 = \text{constant} > 0, \tag{3.12}$$

$$\operatorname{trace}(RH_3 - H_3R)^2 = 32a^4e^2 = 2a^2K_N, \tag{3.13}$$

$$\operatorname{trace}(RH_4 - H_4R)^2 = 2(d^2+e^2)K_N. \tag{3.14}$$

Hence, (3.8) reduces to

$$\tfrac{1}{2}\Delta K_N = 4\sum S_{4ij}^3(h_{ikl}^3h_{jkl}^4 - h_{ikl}^4h_{jkl}^3) + 3(1-S)K_N + \sum (S_{\beta ijk}^{\alpha})^2. \tag{3.15}$$

If $m = 2$, then from (3.12) and (3.15) we obtain

$$K_N(1 - S) = \tfrac{32}{3} \sum_l (a_{,l}e)^2 \geqq 0, \tag{3.16}$$

where $a_{,l}$ denotes the covariant derivative of a. By Lemma 1 of [3], (3.5) and (3.16) we obtain

$$\begin{aligned} \sum h_{ij}^{\alpha}\Delta h_{ij}^{\alpha} &\geqq -S_3^2 - S_4^2 + \tfrac{3}{2}S - 4S_3S_4 \\ &= \tfrac{3}{2}(1 - S)S + \tfrac{1}{2}(S_3 - S_4)^2 \\ &\geqq \tfrac{1}{2}(S_3 - S_4)^2 \geqq 0. \end{aligned} \tag{3.17}$$

Hence we obtain

$$\tfrac{1}{2}\Delta S = \sum (h_{ijk}^{\alpha})^2 + \sum h_{ij}^{\alpha}\Delta h_{ij}^{\alpha} \geqq 0. \tag{3.18}$$

By Hopf's lemma, we see that S is constant and

$$h_{ijk}^{\alpha} = 0, \tag{3.19}$$

$$\sum h_{ij}^{\alpha}\Delta h_{ij}^{\alpha} = 0. \tag{3.20}$$

Formula (3.20) implies

$$N(H_3H_4 - H_4H_3)^2 = 2N(H_3)N(H_4), \tag{3.21}$$

$$S_3 = S_4 = \tfrac{1}{2}. \tag{3.22}$$

Therefore, by Lemma 1 of [3] we see that H_3 and H_4 are scalar multiples of

$$\tilde{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \tilde{B} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

respectively. Hence, we may conclude that $d=0$ and

$$H_3 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}, \qquad H_4 = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}. \tag{3.23}$$

On the other hand, since $e_2 = Je_1$, we can easily show that the second fundamental form in the direction of Je_3 is given by

$$\begin{bmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix}.$$

Since $h_{ijk}^{\alpha}=0$, we have

$$dh_{ij}^{\alpha} = \sum h_{ik}^{\alpha}\omega_j^k + \sum h_{kj}^{\alpha}\omega_i^k - \sum h_{ij}^{\beta}\omega_{\beta}^{\alpha}. \tag{3.24}$$

Using (3.23) and (3.24) we can prove that

$$\omega_3^4 = 2\omega_1^2. \tag{3.25}$$

Therefore, we may conclude that M is a complex quadric in CP^2.

Case $K_N=0$. This case is contained in the following:

PROPOSITION. *Let M be a complex submanifold of a Kähler manifold $\tilde{M}$ with complex structure J. Then the scalar normal curvature $K_N=0$ at $p\in M$ if and only if the second fundamental form $\boldsymbol{h}=0$ at p.*

PROOF. Let A be the second fundamental form in the direction $\boldsymbol{e}$. Then if $K_N=0$ at p, we have

$$N(AJA - JAA) = 0.$$

This implies that

$$\operatorname{trace}\,(AJA^t(AJA)) = \operatorname{trace} A^4 = 0. \tag{3.26}$$

By (3.26) and $A={}^tA$, we obtain $A=0$ at p. Consequently, $\boldsymbol{h}=0$ at p. The converse of this is trivial. This proves the proposition. Hence the Main Theorem is proved completely.

REFERENCES

1. B.-Y. Chen, *Pseudo-umbilical submanifolds of a Riemannian manifold of constant curvature*. III, J. Differential Geometry (to appear).

2. B.-Y. Chen and G. D. Ludden, *Surfaces with mean curvature vector parallel in the normal bundle* (to appear).

3. S. S. Chern, M. do Carmo and S. Kobayashi, *Minimal submanifolds of a sphere with second fundamental form of constant length*, Functional Analysis and Related Fields, 1970, pp. 57–75.

4. T. Itoh, *Minimal surfaces in 4-dimensional Riemannian manifold of constant curvature*, Kōdai Math. Sem. Rep. (to appear).

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