SEMIGROUPS CORRESPONDING TO ALGEBROID BRANCHES IN THE PLANE

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Abstract. The symmetric semigroups of nonnegative integers and their generators, corresponding to algebroid branches of the plane, are determined.

Let $\alpha$ be an algebroid branch of a plane curve with coefficients in an algebraically closed field with characteristic 0. Although the semigroup $S(\alpha)$ of $\alpha$ is symmetric, not all symmetric semigroups correspond to branches [1]. Subsequently the nonredundant generators of $S(\alpha)$ are found from which follow necessary and sufficient conditions for a nonredundant set of positive integers to generate the semigroup of a branch $\alpha$. A method for obtaining the generators from the power series is given.

The definitions for infinitely near points $P_j$, multiplicity sequence, proximity structure, satellite cluster and multiplicity matrix are given in [2]. By the restriction of a point $P_j$ we mean the number of points $P_j$ is proximate to, and the leading points are the points which have successors with increased restriction. Defining the order of a divisor $D$ on $\alpha$, $o(D, \alpha)$, in the usual way, let $v(P_j, \alpha) = \min(o(D, \alpha))$, where $\alpha$ passes thru $P_j$ and $D$ is a divisor of $P_j$.

Lemma 1. If $\alpha$ has satellite clusters $S_1, \ldots, S_n$ and $\alpha^*$ is obtained from $\alpha$ by deleting $S_n$, then $v(P_j, \alpha) = \min(v(P_j, \alpha^*))$, $0 \leq j \leq l$, $P_l$ the last satellite point of $\alpha^*$.

Proof. Let $M(\alpha) = (m_{ij})$, $0 \leq j \leq l$, be the upper triangular multiplicity matrix of $\alpha$. Then $v(P_j, \alpha) = \sum_{i=0}^{l} m_{ij}r$, $j \leq t$ [2]. Since $M(\alpha^*) = (m_{ij}^*)$ consists of the first $l$ rows and columns of $M(\alpha)$, by using the proximity structure, for $j \leq l$,

$$
\sum_{i=0}^{l} m_{ij}^* = \sum_{i=0}^{l} m_{ij}^* (m_{ij} m_{ij}^*) = m_{ij}^* v(P_j, \alpha^*).
$$

Lemma 2. If $L_j$ are the leading points, $0 \leq j \leq n$, then

$$
g.c.d. (v(L_0, \alpha), \ldots, v(L_n, \alpha)) = d = 1.
$$

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Proof. If \( a \) has a single satellite cluster, then g.c.d. \((v(L_0, a), v(L_1, a)) = m_{rt} = 1.\) Let \( a \) have \( n \) clusters. Assuming the truth for \( a^* \), defined in Lemma 1, implies \( d|m_{rt} \). Let \( L_{n+1} = P_{r+1} \). Since \( v(P_{r+1}, a) = \sum_{i=0}^n m_{ir} \cdot m_{rt} + m_{r+1,t} \) and since \( m_{tt}|m_{it}, 1 \leq i \leq r, d|m_{rt+1,t} \Rightarrow d|\text{g.c.d.} \ (m_{tt}, m_{r+1,t}) = 1.\)

**Corollary 1.** Let \( a_j \) be the branch whose only satellite cluster is \( S_j \), \( 1 \leq j \leq n \), with multiplicity \( \lambda_j \) at the origin and let \( \lambda_{r+1} = 1.\) Then \( v(L_0, a) = \prod_{i=r+1}^{n+1} \lambda_i \) and g.c.d. \((v(L_0, a), \ldots, v(L_j, a)) = \prod_{i=r+1}^{n+1} \lambda_i.\)

**Lemma 3.** Let \( a \) have a single satellite cluster. Let \( L_1 = P_{r+1} \). Then \( m_r m_{r+1,t} = \sum_{t=r+1}^t (m_{it})^2.\)

Proof. \[
m_{rt} = q \cdot m_{r+1,t} + m_{r+1+a,t} \Rightarrow m_r m_{r+1,t} = \sum_{t=r+1}^r (m_{it})^2 + m_{r+1,t} m_{r+1+a,t}.
\]

Repeating the process with \( m_{r+1,t} \) and \( m_{r+1+a,t} \) proves Lemma 3.

**Lemma 4.** Let \( E_j \) denote the last satellite point of the \( j \)th cluster of a branch \( a \) with \( n \) clusters. Then \( v(E_j, a) = \lambda_j v(L_j, a).\)

Proof. Assume \( j = n \) and \( L_n = P_{r+1}.\) If \( P_q \) is proximate to \( P_{q-1} \) and \( P_{q-k}, \) then \( m_q = 1 \) and \( m_q = m_{s,a-1} + m_{s,a-k}, 1 < q.\) Therefore the proximity structure and Lemma 3 imply

\[
v(E_n, a) = (m_r - m_{r+1,t}) v(P_r, a) + m_{r+1,t} v(P_{r+1}, a) + \sum_{t=r+2}^r (m_{it})^2
\]

\[
= (m_r - m_{r+1,t}) v(P_r, a) + m_{r+1,t} v(P_{r+1}, a) + m_{r+1,t} v(P_{r+1}, a)
\]

\[
= \lambda_n v(L_n, a).
\]

This and Lemma 1 also prove Lemma 4, if \( j \neq n.\)

**Corollary 2.** If \( n \geq 2, \) then \( \sum_{j=1}^{n} (\lambda_j - 1) v(L_j, a) < v(L_n, a).\)

Proof. Induction on \( n.\)

**Lemma 5.** If \( \sum_{j=1}^{n} a_j v(L_j, a) = \sum_{j=0}^{n} b_j v(L_j, a), \) \( a_j \) and \( b_j \) nonnegative integers, and if \( 0 \leq a_j < \lambda_j, \) then \( b_0 = 0 \) and \( a_j = b_j, 1 \leq j \leq n.\)

Proof. For one satellite cluster the statement follows from g.c.d. \((v(L_0, a), v(L_1, a)) = 1.\) Assume \( a \) has \( n \geq 2 \) clusters and assume two different representations

\[
\sum_{j=1}^{n-1} a_j v(L_j, a) + a_n v(L_n, a) = \sum_{j=0}^{n-1} b_j v(L_j, a) + b_n v(L_n, a).
\]
By Corollary 2, $b_n \leq a_n \Rightarrow a_n = b_n$, since $\lambda_n | a_n - b_n \Rightarrow \sum_{i=0}^{n-1} b_i v(L_i, \alpha) / \lambda_n = \sum_{i=0}^{n-1} b_i v(L_i, \alpha) / \lambda_n$, a contradiction if the truth is assumed for $n-1$ clusters.

**Theorem 1.** $S(\alpha)$ is generated nonredundantly by $v(L_j, \alpha)$, $0 \leq j \leq n$.

**Proof.** By Lemma 5, $\sum_{i=1}^{n} a_i v(L_i, \alpha)$, $0 \leq a_i < \lambda_i$, are distinct first elements of $S(\alpha)$ in the congruence classes mod$(v(L_0, \alpha))$, $\prod_{j=1}^{n} \lambda_j = v(L_0, \alpha)$ in number.

Let $S=\{a_0 < a_1 < \cdots < a_n\}$ be a set of nonredundant integers, $n \geq 1$. Let $\lambda_{n+1} =$g.c.d. $(a_0, \cdots, a_n)=1$ and recursively

$$\lambda_j = \text{g.c.d.}(a_0 | \Pi_{j+1}, \cdots, a_{j-1} | \Pi_{j+1}), \quad 2 \leq j \leq n,$$

$$\Pi_{j+1} = \prod_{\lambda_i}^{n+1} \lambda_i, \quad 1 \leq j \leq n.$$ 

Let $\lambda_1=a_0 | \Pi_2$. From Lemmas 1 and 4 then follows

**Theorem 2.** $S$ generates a semigroup $S(\alpha)$ iff $\lambda_j > 1$, $2 \leq j \leq n$, and $\lambda_j a_j < a_{j+1}$, $1 \leq j \leq n-1$.

Let $\alpha$ have representative

$$x_\alpha = u^{n_0}, \quad y_\alpha = \sum_{h=1}^{n_0} a_{h,0} u^{h v_0} + \sum_{h=1}^{n_1} a_{h,1} u^{h (v_1+1)} + \cdots + \sum_{h=1}^{n_{n-1}} a_{h,n-1} u^{h (v_{n-1}+1)} + a_{1,n} u^{v_n} + \sum_{h=1}^{n} a_{h} u^{v_n+h},$$

where $a_{i,j} \neq 0$, $1 \leq j \leq n$, $1 < d_j = \text{g.c.d.}(v_0, \cdots, v_j)$, $1 \leq j \leq n-1$, $1 = d_n = \text{g.c.d.}(v_0, \cdots, v_n)$.

Then $d_j = \prod_{i=j+1}^{n+1} \lambda_i$, $v_0 = \prod_{i=1}^{n+1} \lambda_i$, $\lambda_i$ defined in Corollary 1, $v_j = v(L_j, \alpha)$ and for $2 \leq j \leq n$, $v_j = m(j) + v_{j-1} + \lambda_{j-1} d_j$, where $m(j)$ is the multiplicity of $L_j$ and $\lambda_{j-1} + 1$ is the number of free points following the $(j-1)$st cluster [3, Chapter XI, §6.1]. Hence $v_j = v(L_j, \alpha) + \sum_{i=2}^{j} [m(i)+\lambda_{i-1} d_i]$. By Lemma 4,

$$\lambda_{i-1} v(L_{i-1}, \alpha) + \lambda_{i-1} d_i + m(i) = v(L_i, \alpha)$$

$$\Rightarrow v_j = \sum_{i=1}^{j} v(L_i, \alpha) - \sum_{i=2}^{j} \lambda_{i-1} v(L_{i-1}, \alpha)$$

$$\Rightarrow v(L_j, \alpha) = v_j + \sum_{i=2}^{j} (\lambda_{i-1} - 1) v(L_{i-1}, \alpha),$$

which determines the generators recursively.
References


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