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SEMIGROUPS CORRESPONDING TO ALGEBROID BRANCHES IN THE PLANE

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ABSTRACT. The symmetric semigroups of nonnegative integers and their generators, corresponding to algebroid branches of the plane, are determined.

Let α be an algebroid branch of a plane curve with coefficients in an algebraically closed field with characteristic 0. Although the semigroup $S(\alpha)$ of α is symmetric, not all symmetric semigroups correspond to branches [1]. Subsequently the nonredundant generators of $S(\alpha)$ are found from which follow necessary and sufficient conditions for a nonredundant set of positive integers to generate the semigroup of a branch α. A method for obtaining the generators from the power series is given.

The definitions for infinitely near points P_j, multiplicity sequence, proximity structure, satellite cluster and multiplicity matrix are given in [3]. By the restriction of a point P_j we mean the number of points P_j is proximate to, and the leading points are the points which have successors with increased restriction. Defining the order of a divisor D on α, $o(D, \alpha)$, in the usual way, let $v(P_j, \alpha)=\min(o(D, \alpha))$, where α passes thru P_j and D is a divisor of P_j.

LEMMA 1. *If α has satellite clusters $S_1, \cdots, S_n$ and α^* is obtained from α by deleting S_n, then $v(P_j, \alpha)=cv(P_j, \alpha^*)$, $0\leqq j\leqq l$, P_l the last satellite point of α^*.*

PROOF. Let $M(\alpha)=(m_{ij})$, $0\leqq j\leqq t$, be the upper triangular multiplicity matrix of α. Then $v(P_j, \alpha)=\sum_{i=0}^{t} m_{ij}m_{it}$, $j\leqq t$ [2]. Since $M(\alpha^*)=(m^*_{ij})$ consists of the first l rows and columns of $M(\alpha)$, by using the proximity structure, for $j\leqq l$,

$$\sum_{i=0}^{t} m_{ij}m_{it} = \sum_{i=0}^{l} m^*_{ij}(m_{lt}m^*_{il}) = m_{lt}v(P_j, \alpha^*).$$

LEMMA 2. *If L_j are the leading points, $0\leqq j\leqq n$, then*

$$\text{g.c.d. } (v(L_0, \alpha), \cdots, v(L_n, \alpha)) = d = 1.$$

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PROOF. If α has a single satellite cluster, then g.c.d. $(v(L_0, \alpha), v(L_1, \alpha))= m_{tt}=1$. Let α have n clusters. Assuming the truth for α^*, defined in Lemma 1, implies $d|m_{lt}$. Let $L_{n+1}=P_{r+1}$. Since $v(P_{r+1}, \alpha)=\sum_{i=0}^{r} m_{ir} \cdot m_{it}+m_{r+1,t}$ and since $m_{lt}|m_{it}$, $1\leqq i\leqq r$, $d|m_{r+1,t}\Rightarrow d|$g.c.d. $(m_{lt}, m_{r+1,t})=1$.

COROLLARY 1. *Let α_j be the branch whose only satellite cluster is S_j, $1\leqq j\leqq n$, with multiplicity λ_j at the origin and let $\lambda_{n+1}=1$. Then $v(L_0, \alpha)= \prod_{i=1}^{n+1} \lambda_i$ and* g.c.d. $(v(L_0, \alpha), \cdots, v(L_j, \alpha))= \prod_{i=j+1}^{n+1} \lambda_i$.

LEMMA 3. *Let α have a single satellite cluster. Let $L_1=P_{r+1}$. Then $m_{rt}m_{r+1,t}=\sum_{i=r+1}^{t} (m_{it})^2$.*

PROOF.

$$\begin{aligned} m_{rt} &= q \cdot m_{r+1,t} + m_{r+1+q,t} \\ &\Rightarrow m_{rt}m_{r+1,t} = \sum_{i=r+1}^{r+q} (m_{it})^2 + m_{r+1,t}m_{r+1+q,t}. \end{aligned}$$

Repeating the process with $m_{r+1,t}$ and $m_{r+1+q,t}$ proves Lemma 3.

LEMMA 4. *Let E_j denote the last satellite point of the jth cluster of a branch α with n clusters. Then $v(E_j, \alpha)=\lambda_j v(L_j, \alpha)$.*

PROOF. Assume $j=n$ and $L_n=P_{r+1}$. If P_q is proximate to P_{q-1} and P_{q-k}, then $m_{qq}=1$ and $m_{sq}=m_{s,q-1}+m_{s,q-k}$, $s<q$. Therefore the proximity structure and Lemma 3 imply

$$\begin{aligned} v(E_n, \alpha) &= (m_{rt} - m_{r+t,t})v(P_r, \alpha) + m_{r+1,t}v(P_{r+1}, \alpha) + \sum_{i=r+2}^{t} (m_{it})^2 \\ &= (m_{rt} - m_{r+1,t})(v(P_r, \alpha) + m_{r+1,t}) + m_{r+1,t}v(P_{r+1}, \alpha) \\ &= \lambda_n v(L_n, \alpha). \end{aligned}$$

This and Lemma 1 also prove Lemma 4, if $j\neq n$.

COROLLARY 2. *If $n\geqq 2$, then $\sum_{j=1}^{n-1} (\lambda_j-1)v(L_j, \alpha)<v(L_n, \alpha)$.*

PROOF. Induction on n.

LEMMA 5. *If $\sum_{j=1}^{n} a_j v(L_j, \alpha)=\sum_{j=0}^{n} b_j v(L_j, \alpha)$, a_j and b_j nonnegative integers, and if $0\leqq a_j<\lambda_j$, then $b_0=0$ and $a_j=b_j$, $1\leqq j\leqq n$.*

PROOF. For one satellite cluster the statement follows from g.c.d. $(v(L_0, \alpha), v(L_1, \alpha))=1$. Assume α has $n\geqq 2$ clusters and assume two different representations

$$\sum_{j=1}^{n-1} a_j v(L_j, \alpha) + a_n v(L_n, \alpha) = \sum_{j=0}^{n-1} b_j v(L_j, \alpha) + b_n v(L_n, \alpha).$$

By Corollary 2, $b_n \leqq a_n \Rightarrow a_n = b_n$, since $\lambda_n | a_n - b_n \Rightarrow \sum_{j=1}^{n-1} a_j(v(L_j, \alpha)/\lambda_n) = \sum_{j=0}^{n-1} b_j(v(L_j, \alpha)/\lambda_n)$, a contradiction if the truth is assumed for $n-1$ clusters.

THEOREM 1. *$S(\alpha)$ is generated nonredundantly by $v(L_j, \alpha)$, $0 \leqq j \leqq n$.*

PROOF. By Lemma 5, $\sum_{j=1}^{n} a_j v(L_j, \alpha)$, $0 \leqq a_j < \lambda_j$, are distinct first elements of $S(\alpha)$ in the congruence classes $\operatorname{mod}(v(L_0, \alpha))$, $\prod_{j=1}^{n} \lambda_j = v(L_0, \alpha)$ in number.

Let $S = \{a_0 < a_1 < \cdots < a_n\}$ be a set of nonredundant integers, $n \geqq 1$. Let $\lambda_{n+1} = \text{g.c.d. } (a_0, \cdots, a_n) = 1$ and recursively

$$\lambda_j = \text{g.c.d.}(a_0 \,|\, \Pi_{j+1}, \cdots, a_{j-1} \,|\, \Pi_{j+1}), \qquad 2 \leqq j \leqq n,$$

$$\Pi_{j+1} = \prod_{i=j+1}^{n+1} \lambda_i, \qquad 1 \leqq j \leqq n.$$

Let $\lambda_1 = a_0 | \Pi_2$. From Lemmas 1 and 4 then follows

THEOREM 2. *S generates a semigroup $S(\alpha)$ iff $\lambda_j > 1$, $2 \leqq j \leqq n$, and $\lambda_j a_j < a_{j+1}$, $1 \leqq j \leqq n-1$.*

Let α have representative

$$x_\alpha = u^{\nu_0}, \qquad y_\alpha = \sum_{h=1}^{g(0)} a_{h,0} u^{h \cdot \nu_0} + \sum_{h=1}^{g(1)} a_{h,1} u^{\nu_1 + (h-1)d_1}$$
$$+ \cdots + \sum_{h=1}^{g(n-1)} a_{h,n-1} u^{\nu_{n-1} + (h-1)d_{n-1}} + a_{1,n} u^{\nu_n} + \sum_{h=1}^{\infty} a_h u^{\nu_n + h},$$

where $a_{1,j} \neq 0$, $1 \leqq j \leqq n$, $1 < d_j = \text{g.c.d.}(\nu_0, \cdots, \nu_j)$, $1 \leqq j \leqq n-1$, $1 = d_n = \text{g.c.d.}(\nu_0, \cdots, \nu_n)$.

Then $d_j = \prod_{i=j+1}^{n+1} \lambda_i$, $\nu_0 = \prod_{i=1}^{n+1} \lambda_i$, λ_i defined in Corollary 1, $\nu_1 = v(L_1, \alpha)$ and for $2 \leqq j \leqq n$, $\nu_j = m(j) + \nu_{j-1} + k_{j-1} d_j$, where $m(j)$ is the multiplicity of L_j and $k_{j-1}+1$ is the number of free points following the $(j-1)$st cluster [**3**, Chapter XI, §6.1]. Hence $\nu_j = v(L_1, \alpha) + \sum_{i=2}^{j} [m(i) + k_{i-1} d_i]$. By Lemma 4,

$$\lambda_{i-1} v(L_{i-1}, \alpha) + k_{i-1} d_i + m(i) = v(L_i, \alpha)$$
$$\Rightarrow \nu_j = \sum_{i=1}^{j} v(L_i, \alpha) - \sum_{i=2}^{j} \lambda_{i-1} v(L_{i-1}, \alpha)$$
$$\Rightarrow v(L_j, \alpha) = \nu_j + \sum_{i=2}^{j} (\lambda_{i-1} - 1) v(L_{i-1}, \alpha),$$

which determines the generators recursively.

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