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A STRONG MAXIMUM PRINCIPLE FOR QUASILINEAR PARABOLIC DIFFERENTIAL INEQUALITIES

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ABSTRACT. A maximum principle for C^1 solutions of quasilinear parabolic differential inequalities which retains the strong conclusion of Nirenberg's well-known result [2] is established.

The case of strongly differentiable solutions rather than of class C^1 is also discussed.

1. **Introduction. Statement of results.** Let Q be a domain in an $(n+1)$-dimensional space $(x, t)\equiv(x_1, \cdots, x_n, t)$.

In [2], L. Nirenberg derived a strong maximum principle for second order parabolic operators in Q with continuous coefficients. This result, as it is pointed out in [2], can be carried over to nonlinear parabolic equations of the form

$$\Phi(x, t, u, Du, D^2u) = u_t,$$

where $Du=(\partial u/\partial x_1, \cdots, \partial u/\partial x_n)$ denotes the spatial gradient and D^2u the Hessian matrix of the dependent variable $u=u(x, t)$. Moreover it is assumed that the function u has continuous first and second derivatives with respect to the x_i and continuous first derivative with respect to t.

The purpose of this paper is to find a maximum principle which applies to C^1 solutions in Q of parabolic differential inequalities and still has the strong conclusion of Nirenberg's result.

In this way we extend to a time-dependent situation an interesting maximum principle for C^1 solutions of elliptic differential inequalities recently established by J. Serrin in [3], that retains the strong conclusion of the well-known Hopf result [1].

We shall consider the second order quasilinear partial differential inequalities

$$\begin{aligned} &\operatorname{div} A(x, t, u, Du) - B(x, t, u, Du) \geqq u_t, \\ &\operatorname{div} A(x, t, v, Dv) - B(x, t, v, Dv) \leqq v_t, \end{aligned} \tag{1}$$

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where $A=(A_1, \cdots, A_n)$ is a given vector function of (x, t, u, Du) and B is a given scalar function of the same variables. By div $A(x, t, u, Du)$ we mean the divergence, with respect to the variable x, of $A(x, t, u(x, t), Du(x, t))$.

As to the structure of the functions $A(x, t, u, p)$ and $B(x, t, u, p)$ we shall assume the following:

(i) *the function $A(x, t, u, p)$ is differentiable with respect to the variables u and p*;

(ii) *the function $B(x, t, u, p)$ is Lipschitz continuous in the variables u and p*.

Let $Q_\tau = Q \cap \{t=\tau\}$. Since for the main result we assume only that the functions u and v are of class C^1 in Q, we need to interpret (1) in the weak form:

$$\text{(2)} \qquad \begin{aligned} &\int_Q \{\phi u_t + D\phi \cdot A(x, t, u, Du) + \phi B(x, t, u, Du)\}\, dx\, dt \leqq 0, \\ &\int_Q \{\phi v_t + D\phi \cdot A(x, t, v, Dv) + \phi B(x, t, v, Dv)\}\, dx\, dt \geqq 0, \end{aligned}$$

valid for all nonnegative $\phi \in C^1(Q)$, having compact support in Q_t, for each t.

To state our result it is convenient, for each point $(x_0, t_0) \in Q$, to denote by $S_{(x_0,t_0)}$ *the set of all points $(x, t) \in Q$ which can be joined to (x_0, t_0) by a downward directed curve, with (x_0, t_0) as initial point and (x, t) as endpoint, namely by a curve given by the equations*

$$x_i = \varphi_i(t) \qquad (i = 1, 2, \cdots, n),$$

where φ_i are of class C^0 on $[t, t_0]$, $t < t_0$.

We shall prove the following:

THEOREM 1. *Let $u=u(x, t)$ and $v=v(x, t)$ be functions of class C^1 on Q. Suppose that $u=v$ at some point $(x_0, t_0) \in Q$ and that $u \leqq v$ in $S_{(x_0,t_0)}$. Assume further that u and v satisfy the respective differential inequalities* (1) *in $S_{(x_0,t_0)}$, where at least one of the matrices*

$$\partial A/\partial(Du) \quad \text{or} \quad \partial A/\partial(Dv)$$

is to be supposed positive definite. Then $u \equiv v$ in $S_{(x_0,t_0)}$.

REMARK 1. Let us introduce the notation

$$\text{(3)} \qquad K_{x_0}(\rho) \times (t_0 - \tau\rho^2, t_0),$$

where $K_{x_0}(\rho)$ stands for a cube of side ρ and center x_0, to describe a space-time *rectangular neighborhood* of (x_0, t_0) on an $(n+1)$-dimensional space.

What essentially we shall prove is that, if the equality $u=v$ holds at a single point (x_0, t_0), under the assumptions of the above theorem, then it holds everywhere on a rectangular neighborhood of (x_0, t_0) of the form (3) contained in Q. From this fact then it shall follow that the equality $u=v$ holds in $S_{(x_0,t_0)}$.

Actually, as it is pointed out later after the proof of Theorem 1, the same result is valid under a weaker assumption on the function $A(x, t, u, p)$. More precisely we can assume that

(i′) *the function $A(x, t, u, p)$ is Lipschitz continuous in the variable u and differentiable with respect to the variable p.*

Next, to turn to solutions of (1) of class $W^{1,2}_{\mathrm{loc}}$ in Q we shall interpret (1) in the weak sense, that is in the sense that (2) are required to hold for all bounded nonnegative functions $\phi \in W^{1,2}(Q)$, having compact support in Q_t, for each t.[2]

In this case if we assume that

(iii) *the functions $A(x, t, u, p)$ and $B(x, t, u, p)$ are uniformly Lipschitz continuous with respect to the variables u and p,*

we are able to prove the following:

THEOREM 2. *Let $u=u(x, t)$ and $v=v(x, t)$ be continuous functions in Q, having strong derivatives with respect to x and t of class L^2_{loc}. Suppose that $u=v$ at some point $(x_0, t_0)\in Q$ and that $u \leqq v$ in $S_{(x_0,t_0)}$. Assume further that u and v satisfy the respective differential inequalities (1) in $S_{(x_0,t_0)}$, where the matrix $\partial A/\partial p$ is to be supposed uniformly positive definite. Then $u\equiv v$ in $S_{(x_0,t_0)}$.*

Clearly the continuity assumption on u and v can be dropped provided that equalities and inequalities are considered in the almost everywhere sense, without essentially changing the proof.

The assumption on the matrix $\partial A/\partial p$ tells that the operator $\operatorname{div} A(x, t, u, Du)$ is uniformly elliptic, but it is worth emphasizing that such a condition is not required to state the main result.

To carry out the proofs we shall follow the outline of Serrin's proof for the elliptic case **[3]**; here we shall invoke and suitably apply a remarkable time-dependent Harnack-type inequality proved by N. Trudinger in **[4]**.

2. Proof of results.

PROOF OF THEOREM 1. The main goal of the proof is to show that $u\equiv v$ on a rectangular neighborhood of (x_0, t_0), as described in Remark 1, contained in Q.

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[2] $W^{1,2}(Q)$ is the Sobolev space of all square integrable functions in Q, which have square integrable first derivatives with respect to x and t. The derivatives are to be taken in the sense of distributions theory. $W^{1,2}(Q)$ is a Hilbert space under the norm $\|u\|=(\int_Q (u^2+\sum_1^n u_{x_i}^2+u_t^2)\,dx\,dt)^{1/2}$. We define the local space $W^{1,2}_{\mathrm{loc}}(Q)=\{u(x,t): u, Du, u_t \in L^2_{\mathrm{loc}}(Q)\}$.

To this end let R denote the rectangular neighborhood of (x_0, t_0),

$$R = K_{x_0}(\rho) \times (t_0 - \tau\rho^2, t_0), \tag{3}$$

ρ and τ being chosen so small that R is contained in Q.

We notice that as a matter of fact in the inequalities (2) we may restrict consideration to nonnegative functions ϕ which vanish outside R and correspondingly the integrations can take place over R. So, as can readily be seen by subtracting the first inequality of (2) from the second, the following important relation holds:

$$\begin{aligned}\int_R \{\phi(v_t - u_t) + D\phi\cdot[A(x, t, v, Dv) - A(x, t, u, Du)]& \\ + \phi[B(x, t, v, Dv) - B(x, t, u, Du)]\}\, dx\, dt &\geqq 0\end{aligned} \tag{4}$$

for all nonnegative functions $\phi\in C^1(R)$, which vanish outside R_t, for each t $(R_\tau = R\cap\{t=\tau\})$.

The hypotheses (i) and (ii) yield in R

$$\begin{aligned}|A(x, t, v, Dv) - A(x, t, u, Du)| &\leqq a\,|Dw| + b\,|w|,\\ |B(x, t, v, Dv) - B(x, t, u, Du)| &\leqq c\,|Dw| + d\,|w|,\end{aligned} \tag{5}$$

where w denotes the function difference $v-u$ and a, b, c, d are suitable constants depending on the structure of the functions A and B as well as on bounds for u, v, Du and Dv in R.

Let the matrix $\partial A/\partial(Du)$ be assumed positive definite (in the other case the proof is carried out similarly). Then, in view of the continuity of the matrix $\partial A/\partial(Du)$, there exists a positive constant λ such that, in R, $\partial A/\partial(Du)\geqq\lambda$.

As in [**3**], we can easily show that

$$(Dw)\cdot\{A(x, t, v, Dv) - A(x, t, u, Du)\} \geqq \frac{\lambda}{4}|Dw|^2 - \left(\frac{2b^2}{\lambda} + \frac{\lambda}{2}\right)w^2 \tag{6}$$

in $R=K_{x_0}(\rho)\times(t_0-\tau\rho^2, t_0)$ provided that ρ and τ are sufficiently small.

Let us now consider the nonnegative function w as a solution of the differential inequality (4). Clearly the relations (5) and (6) allow us to apply a time-dependent Harnack-type inequality due to N. Trudinger [**4**, Theorem 1.2].

We wish to do this in a suitable manner and precisely applying repeatedly Trudinger's result to couples of subrectangles of R separated by successively smaller nonempty time intervals. For this purpose, let us consider the following, respectively increasing and decreasing, sequences $\{R_\nu^*\}$ and

$\{R_\nu^-\}$ of subrectangles of R:

$$R_\nu^* = K_{x_0}(\rho'') \times \left(t_0 - \tau\rho^2, t_0 - \frac{1}{\nu}\rho^2\right),$$

$$R_\nu^- = K_{x_0}(\rho') \times \left(t_0 - \frac{1}{\nu+1}\rho^2, t_0\right),$$

where $0<\rho'<\rho''<\rho$ and the integer ν is such that $\nu \geqq [1/\tau]+1$.

Thus, in view of the aforementioned Harnack-type inequality, we have

$$\int_{t_0-\tau\rho^2}^{t_0-\rho^2/\nu} \int_{K_{x_0}(\rho'')} w(x, t)\, dx\, dt \leqq C\rho^{n+2} \cdot \min_{R_\nu^-} w(x, t), \tag{7}$$

for $\nu = [1/\tau]+1, [1/\tau]+2, \cdots$, where C depends on the structure and on the geometric constant ρ'/ρ, ν and τ.

On the other hand, since w is nonnegative on R and $w(x_0, t_0)=0$, the right-hand side of (7) is zero for any ν and therefore we get

$$\int_{t_0-\tau\rho^2}^{t_0-\rho^2/\nu} \int_{K_{x_0}(\rho'')} w(x, t)\, dx\, dt \leqq 0,$$

for $\nu = [1/\tau]+1, [1/\tau]+2, \cdots$.

Letting $\nu \to \infty$ then yields

$$\int_{t_0-\tau\rho^2}^{t_0} \int_{K_{x_0}(\rho'')} w(x, t)\, dx\, dt \leqq 0.$$

Hence $w \equiv 0$, namely $u \equiv v$, in the rectangular neighborhood of (x_0, t_0)

$$K_{x_0}(\rho'') \times (t_0 - \tau\rho^2, t_0),$$

with τ and ρ determined as above and $0<\rho''<\rho$.

Now the conclusion of the theorem, which is that $u \equiv v$ in $S_{(x_0,t_0)}$, follows by a standard argument.

REMARK 2. The same result holds even if the function $A(x, t, u, p)$ is assumed to satisfy instead of the condition (i) the weaker one (i′).

As a matter of fact, in this case the proof can be carried out the same as before, except that when we consider the difference $A(x, t, v, Dv) - A(x, t, u, Du)$ it will be more convenient to write it as follows

$$\begin{aligned} &A(x, t, v, Dv) - A(x, t, u, Du) \\ &\quad = [A(x, t, u, Dv) - A(x, t, u, Du)] + [A(x, t, v, Dv) - A(x, t, u, Dv)] \end{aligned}$$

and then to estimate separately.

PROOF OF THEOREM 2. The argument is exactly the same as that of Theorem 1. As there, estimates (5) hold in the rectangular neighborhood

R of (x_0, t_0). Furthermore, condition (6) is still valid in R; in order to check this we shall make use of the remark above and take into account the fact that the matrix $\partial A/\partial p$ is uniformly positive definite. Therefore, we may apply, suitably as in the proof of Theorem 1, Trudinger's result to the function $w=v-u$. This leads to finding a suitable rectangular neighborhood of (x_0, t_0) where $w\equiv 0$. Again that is enough to complete the proof of the theorem.

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