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THE BOLD STRATEGY IN PRESENCE OF HOUSE LIMIT

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ABSTRACT. It is known that an optimal strategy for a gambler, who wishes to maximize the probability of winning an amount $a-x$ in a subfair red-and-black casino if his initial capital is x, is the bold strategy in which the gambler wagers at each opportunity the minimum of his entire current capital x' and the amount $a-x'$ required to reach the goal a if he wins the bet. If the casino imposes an upper limit L on wagers, we shall prove that the modified bold strategy of wagering $\min(x', a-x', L)$ is optimal, at least in the important special case in which the goal a is an integral multiple of the house limit L.

1. **The probability of success using the modified bold strategy.** Let $P(x)$ be the probability of reaching the goal a from an initial capital x in a red-and-black casino [**1**, Chapter 5], which does not permit wagers exceeding L, when the gambler uses the modified bold strategy of wagering $\min(x', a-x', L)$ when his current capital is x'. If p is the probability of winning an individual game and $q=1-p$ (so that $0<p<1$, $0<q<1$), it is clear that

$$P(x) = pP(2x), \qquad 0 \leqq x < L, \tag{1}$$

$$P(x) = pP(x+L) + qP(x-L), \qquad L \leqq x < a-L, \tag{2}$$

$$P(x) = p + qP(2x-a), \qquad a-L \leqq x < a. \tag{3}$$

We assume that $a=nL$ for some integer n greater than 2. (If $n=1$ or 2, the ordinary bold strategy involves no wagers greater than L.)

For each x there exist a unique integer m (the quotient) and a unique number R (the remainder) such that $x=mL+R$, $0\leqq R<L$.

LEMMA 1. *The function*

$$P(x) = \frac{[1+(u-1)Q(R/L)]u^m - 1}{u^n - 1}, \qquad u = q/p, \tag{4}$$

satisfies equations (1), (2), *and* (3) *if, and only if, the function* $Q(f)$ *satisfies*

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the equations

$$Q(f) = pQ(2f), \qquad 0 \leqq f < \tfrac{1}{2}, \tag{5}$$

$$Q(f) = p + qQ(2f - 1), \qquad \tfrac{1}{2} \leqq f < 1. \tag{6}$$

We start the proof of this lemma by constructing Table 1 below to record the quotients and remainders of x, $2x$, $x+L$, $x-L$, and $2x-a$ in each of six mutually exclusive intervals whose union is the interval $[0, a]$. Using the entries in this table, simple algebraic manipulations suffice to establish the lemma.

TABLE 1. Values of the quotients and remainders for the arguments of P in equations (1), (2), and (3).

Interval	Quotient, Remainder x	$2x$	$x \pm L$	$2x - a$
$[0, L/2)$	$0, x$	$0, 2x$		
$[L/2, L)$	$0, x$	$1, 2x - L$		
$[L, a - L)$	m, R		$m \pm 1, R$	
$[a - L, a - L/2)$	$n - 1, L - a + x$			$n - 2, 2(L - a + x)$
$[a - L/2, a)$	$n - 1, L - a + x$			$n - 1, L - 2a + 2x$
$[a, a]$	$n, 0$			

It is known [**2**] that there is a unique bounded function $Q(f)$ on $[0, 1)$ which satisfies equations (5) and (6). In terms of the binary representation of f, $f = \sum_{k=1}^{\infty} f_k 2^{-k}$, $f_k = 0$ or 1, this bounded function is

$$Q(f) = \sum_{k=1}^{\infty} f_k p^k \prod_{j=1}^{k-1} u^{f_j}.$$

If $Q(1)$ is defined as 1, the function $Q(f)$ is continuous and strictly increasing on $[0, 1]$. In addition $Q(f) = f$ if $p = q$, and [**3**] $Q(f)$ is singular (with respect to Lebesgue measure) if $p \neq q$, and $Q(f)$ satisfies a Hölder condition of order $-\log_2 \max(p, q)$ on $[0, 1]$.

Since the desired solution $P(x)$ of equations (1), (2) and (3) is a probability, it is bounded between 0 and 1. In view of the probabilistic interpretation [**1**, p. 85] of $Q(f)$ and the behavior of m and R as functions of x, the following theorem is now obvious.

THEOREM 1. *The probability $P(x)$ is given by equation* (4), *in which $Q(R/L)$ is the probability that a gambler with initial capital R will achieve the goal L, if he uses the bold strategy. Moreover*, $P(0)=0$, $P(a)=1$, *$P(x)$ is continuous and strictly increasing on* $[0, a]$, $P(x)=x/a$ *if $p=q$, and $P(x)$ is singular if $p \neq q$.*

If x is a multiple of L, so that $R=0$, every wager in the modified bold strategy is L, and equation (4) reduces to the classical result [**4**, p. 314] $(u^m - 1)/(u^n - 1) = 1 - (u^n - u^m)(u^n - 1)$.

2. **Optimality of the modified bold strategy.** We shall prove the following result.

THEOREM 2. *The modified bold strategy is optimal in a subfair* ($p<q$) *red-and-black casino with house limit, i.e., no strategy has greater probability of success, when the goal a is an integral multiple of the house limit L.*

In view of [**1**, Theorem 2.12.1], it is sufficient to show that

$$P(x) - pP(x+w) - qP(x-w) \geqq 0 \tag{7}$$

when $0 \leqq w \leqq \min(x, a-x, L)$. If $w=L$, then $x \geqq L$, $a-x \geqq L$ and the left-hand side of the inequality (7) vanishes by virtue of equation (2). Hence we may assume that $w<L$. We tabulate in Table 2 the quotient and remainder for $x+w$ and $x-w$ in each of four mutually exclusive triangles whose union is the square $0 \leqq w < L$, $0 \leqq R < L$.

TABLE 2. Values of the quotients and remainders for the arguments $x \pm w$ of P in the inequality (7).

Triangle	Quotient, Remainder $x+w$	$x-w$
$0 \leqq R < w < L-R$	$m, R+w$	$m-1, L+R-w$
$L-w \leqq R < w < L$	$m+1, R+w-L$	$m-1, L+R-w$
$L-R \leqq w \leqq R < L$	$m+1, R+w-L$	$m, R-w$
$0 \leqq w \leqq R < L-w$	$m, R+w$	$m, R-w$

Using the entries in this Table and equation (4), we see that the function $P(x)$ will satisfy the inequality (7) if, and only if, the function $Q(f)$ satisfies the following inequalities:

$$Q(f) \geqq p[Q(f+g) + Q(f+1-g) - 1], \qquad 0 \leqq f < g < 1-f, \tag{8}$$

$$Q(f) \geqq pQ(f+1-g) + qQ(f-1+g), \qquad 1-g \leqq f < g < 1, \tag{9}$$

$$Q(f) \geqq p + q[Q(f-g) + Q(f-1+g)], \qquad 1-f \leqq g \leqq f < 1, \tag{10}$$

$$Q(f) \geqq pQ(f+g) + qQ(f-g), \qquad 0 \leqq g \leqq f < 1-g. \tag{11}$$

In these inequalities, $f=R/L$ and $g=w/L$.

It is known [**1**, pp. 87–89] that the inequality (11) holds on the closed triangle $0 \leqq g \leqq f \leqq 1-g$, and (9) is obtained from (11) by replacing g by $1-g$. Moreover, the replacement in (11) of f by $f+\frac{1}{2}$ and of g by $g-\frac{1}{2}$ and $\frac{1}{2}-g$ shows that

$$Q(f+\tfrac{1}{2}) \geqq pQ(f+g) + qQ(f+1-g), \qquad \tfrac{1}{2} \leqq g \leqq 1-f \leqq 1, \tag{12}$$

$$Q(f+\tfrac{1}{2}) \geqq pQ(f+1-g) + qQ(f+g), \qquad 0 \leqq f \leqq g \leqq \tfrac{1}{2}. \tag{13}$$

Because $q>p$, it follows from (12) if $g \geqq \frac{1}{2}$ and from (13) if $g \leqq \frac{1}{2}$ that

$$Q(f+\tfrac{1}{2}) \geqq q[Q(f+g)+Q(f+1-g)]+p-q, \qquad 0 \leqq f \leqq g \leqq 1-f.$$

In view of equation (2), $Q(f+\frac{1}{2}) = p + qQ(2f)$, and so

$$Q(2f) \geqq Q(f+g)+Q(f+1-g)-1, \qquad 0 \leqq f \leqq g \leqq 1-f,$$

whence the inequality (8) follows from equation (1). Similarly, replacement in (11) of f by $f-\frac{1}{2}$ and g by $g-\frac{1}{2}$ and $\frac{1}{2}-g$ shows that

$$Q(f-\tfrac{1}{2}) \geqq pQ(f-1+g)+qQ(f-g), \qquad \tfrac{1}{2} \leqq g \leqq f \leqq 1,$$
$$Q(f-\tfrac{1}{2}) \geqq pQ(f-g)+qQ(f-1+g), \qquad 0 \leqq 1-f \leqq g \leqq \tfrac{1}{2},$$

and consequently, since $q>p$,

$$Q(f-\tfrac{1}{2}) \geqq pQ(f-g)+pQ(f-1+g), \qquad 1-f \leqq g \leqq f \leqq 1.$$

Using equation (1) we see that

$$Q(2f-1) \geqq Q(f-g)+Q(f-1+g), \qquad 1-f \leqq g \leqq f \leqq 1,$$

and now the inequality (10) follows from equation (2). This completes the proof of the inequality (7) and hence of Theorem 2.

We can write equation (4) in the form

$$P(x) = Q(R/L)\left\{\frac{u^{m+1}-1}{u^n-1}\right\} + [1-Q(R/L)]\left\{\frac{u^m-1}{u^n-1}\right\},$$

and interpret this result as the probability of success using the following strategy. Wager the remainder R boldly in an attempt to achieve the goal L. Whether successful or not, continue with constant wagers of L. Therefore, the optimal strategy is not unique.

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