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CENTRALIZERS OF THE FOURIER ALGEBRA OF AN AMENABLE GROUP

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ABSTRACT. Let G be a locally compact group with Fourier algebra $A(G)$. We prove that if G is amenable then every centralizer of $A(G)$ is determined by multiplication with an element of the Fourier-Stieltjes algebra of G. This result is then used to show that isometric centralizers correspond to characters of G.

1. **Introduction.** Let A be a Banach algebra. A centralizer of A is a bounded linear operator T on A which commutes with all the operations of left multiplication, i.e.

$$T(xy) = xTy \quad \text{for all } x, y \in A.$$

If G is a locally compact group, denote by $L_1(G)$ the group algebra of G and by $M(G)$ the algebra of bounded Radon measures. In **[6]** Wendel proved that every centralizer of $L_1(G)$ is a convolution with an element of $M(G)$, i.e. there exists a unique measure $\mu \in M(G)$ such that $Tx = x * \mu$ for all $x \in L_1(G)$.

Suppose for the moment that G is Abelian with dual group $\hat{G}$. The set of functions on G which are the Fourier transforms of functions in $L_1(\hat{G})$ is the Fourier algebra $A(G)$. Similarly by considering the transforms of elements in $M(\hat{G})$ we obtain the Fourier-Stieltjes algebra $B(G)$ (see e.g. **[5**, Chapter 1]). Wendel's result now shows that every centralizer of $A(G)$ is given by multiplication with an element of $B(G)$. The purpose of this note is to extend this result to noncommutative groups, the point being that although the dual group no longer exists, the algebras $A(G)$ and $B(G)$ may still be defined.

2. **Definitions and notation.** Throughout we shall employ the notation of **[2]**. Let G be a (not necessarily commutative) locally compact group with identity element e and let $L_1(G)$ denote the group algebra of G. The completion of $L_1(G)$ in the minimal regular norm is the group C^*-algebra, $C^*(G)$. Let $B(G)$ denote the space of all finite linear combinations of continuous positive definite functions on G. Regarded as the dual of $C^*(G)$,

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$B(G)$ is a Banach space and even a commutative Banach algebra under pointwise multiplication. Denote by $A(G)$ the closed subspace generated by elements of $B(G)$ with compact support. It is easy to show that $A(G)$ is an ideal in $B(G)$.

For $u\in B(G)$ and $g\in G$, define ${}_g u$ by ${}_g u(h)=u(gh)$. Then ${}_g u\in B(G)$ and $\|{}_g u\|=\|u\|$. Further if $u\in A(G)$, then so does ${}_g u$.

The algebra $A(G)$ behaves in many ways like a group algebra. In general, however, it lacks one important property, the existence of an approximate identity. In [4] Leptin has shown that an approximate identity exists iff G is amenable (see [3] for various alternative characterizations of amenability). We shall use this result in what follows.

3. **Centralizers of** $A(G)$.

THEOREM 1. *Let G be amenable and let T be a centralizer of $A(G)$. Then there exists a unique element $w\in B(G)$ such that $T=L_w$ (where L_w is defined by $L_w u=uw$ for $u\in A(G)$). Furthermore $\|T\|=\|w\|$.*

PROOF. Let $\{e_\alpha\}$ be an approximate identity for $A(G)$ with $\|e_\alpha\|\equiv 1$. The elements Te_α may be regarded as linear functionals on $C^*(G)$ and by continuity of T, the family $\{Te_\alpha\}$ has a w^*-accumulation point $w\in B(G)$.

Fix $u\in A(G)$. A straightforward density argument shows that uw is a w^*-accumulation point of $\{uTe_\alpha\}$. But then by continuity of T,

$$Tu = \lim_\alpha T(ue_\alpha) = \lim_\alpha uTe_\alpha.$$

This shows that $Tu=uw$, i.e., $T=L_w$.

Since for any $g\in G$ there exists some $u\in A(G)$ with $u(g)\neq 0$, it follows that w is unique. That $\|T\|=\|w\|$ follows from the existence of an approximate identity. Alternately it may be obtained at once from a result of Derighetti [1, Theorem 9].

COROLLARY 1. *In the strong operator topology on $A(G)$, $B(G)$ is a closed subset of the algebra of all bounded operators.*

PROOF. For if $\{w_\alpha\}$ is a directed set in $B(G)$ which converges strongly to an operator S on $A(G)$ then for u, $v\in A(G)$,

$$S(uv) = \lim_\alpha w_\alpha uv = (Su)v$$

so that S is a centralizer. Now apply Theorem 1.

4. **Isometric centralizers.** We consider now the case when a centralizer T is an isometry i.e. $\|Tu\|=\|u\|$ for all $u\in A(G)$. We prove

THEOREM 2. *Let G be amenable and let T be a centralizer of $A(G)$. Then*

T is an isometry iff $T=\lambda L_\kappa$ *where* λ *is some scalar of unit modulus and* κ *is a character of G.*

REMARKS. 1. It is straightforward to show that an operator λL_κ is an isometry so that it suffices to prove the "only if" condition.

2. For some groups G, $A(G)$ may not possess any isometric centralizers apart from those of the form λI since there exist amenable groups which have no nontrivial finite dimensional representations.

By Theorem 1, T is of the form L_w for some $w \in B(G)$. The proof of Theorem 2 depends on the following lemmas.

LEMMA 1. *For all* $g \in G$, $w(g) \neq 0$.

PROOF. Firstly note that if w defines an isometry then so does ${}_g w$ for any g. For if $u \in A(G)$ then

$$\|{}_g w u\| = \|{}_g(w_{g^{-1}} u)\| = \|w_{g^{-1}} u\| = \|{}_{g^{-1}} u\| = \|u\|.$$

Hence it suffices to assume that $w(e)=0$ and to deduce a contradiction.

Choose $u \in A(G)$ with u positive definite and $\|u\|=1$. Then $wu \in A(G)$ and $wu(e)=0$. Choose ϵ such that $0<\epsilon<1$. By [**2**, Lemma 4.13], we may choose $v \in A(G)$ with v positive definite, $\|v\|=1$ and $\|vwu\|<\epsilon$. But then $\epsilon > \|wvu\| = \|vu\| = \|v\| \, \|u\| = 1$ which is a contradiction.

LEMMA 2. L_w *maps* $A(G)$ *onto* $A(G)$.

PROOF. Let N be the range of L_w. Since L_w is an isometry, N is a closed ideal in $A(G)$. By a Tauberian theorem [**2**, Corollary 3.38], it suffices to show that for all $g \in G$ there exists some $v \in N$ with $v(g) \neq 0$. Choose $u \in A(G)$ with $u(g) \neq 0$ and let $v = uw$. The result now follows from Lemma 1.

PROOF OF THEOREM 2. From Lemma 2, L_w^{-1} exists as a bounded operator on $A(G)$. If $u, v \in A(G)$ with $u = u_1 w$ then $L_w^{-1}(uv) = L_w^{-1}(L_w(u_1 v)) = u_1 v = (L_w^{-1} u) v$ so that L_w^{-1} is a centralizer. From this it follows that $w^{-1} \in B(G)$ with $\|w^{-1}\| = 1$.

Now $\|w\|_\infty = \sup_{g \in G} |w(g)| \leqq \|w\| = 1$ and similarly $\|w^{-1}\|_\infty \leqq 1$. Hence $|w(g)| = 1$ for all $g \in G$.

We can write $w(g) = (U_g \xi, \eta)$ where $g \to U_g$ is a unitary representation of G on some Hilbert space H with cyclic vector ξ and with $\|w\| = \|\xi\| \, \|\eta\|$. For all $g \in G$,

$$1 = |w(g)| = |(U_g \xi, \eta)| \leqq \|\xi\| \, \|\eta\| = 1$$

so that by the Cauchy-Schwarz inequality, H is one dimensional and there exist scalars λ_g of unit modulus such that $U_g \xi = \lambda_g \eta$. Let $\lambda = \lambda_e^{-1}$. Then if $\kappa(g) = \lambda^{-1} \lambda_g$, we have that κ is a character of G and $w = \lambda \kappa$. This proves the theorem.

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