CENTRALIZERS OF THE FOURIER ALGEBRA
OF AN AMENABLE GROUP

P. F. RENAUD

Abstract. Let $G$ be a locally compact group with Fourier algebra $A(G)$. We prove that if $G$ is amenable then every centralizer of $A(G)$ is determined by multiplication with an element of the Fourier-Stieltjes algebra of $G$. This result is then used to show that isometric centralizers correspond to characters of $G$.

1. Introduction. Let $A$ be a Banach algebra. A centralizer of $A$ is a bounded linear operator $T$ on $A$ which commutes with all the operations of left multiplication, i.e.

$$T(xy) = xTy \quad \text{for all } x, y \in A.$$  

If $G$ is a locally compact group, denote by $L_1(G)$ the group algebra of $G$ and by $M(G)$ the algebra of bounded Radon measures. In [6] Wendel proved that every centralizer of $L_1(G)$ is a convolution with an element of $M(G)$, i.e. there exists a unique measure $\mu \in M(G)$ such that $Tx = x \ast \mu$ for all $x \in L_1(G)$.

Suppose for the moment that $G$ is Abelian with dual group $\hat{G}$. The set of functions on $G$ which are the Fourier transforms of functions in $L_1(G)$ is the Fourier algebra $A(G)$. Similarly by considering the transforms of elements in $M(\hat{G})$ we obtain the Fourier-Stieltjes algebra $B(G)$ (see e.g. [5, Chapter 1]). Wendel’s result now shows that every centralizer of $A(G)$ is given by multiplication with an element of $B(G)$. The purpose of this note is to extend this result to noncommutative groups, the point being that although the dual group no longer exists, the algebras $A(G)$ and $B(G)$ may still be defined.

2. Definitions and notation. Throughout we shall employ the notation of [2]. Let $G$ be a (not necessarily commutative) locally compact group with identity element $e$ and let $L_1(G)$ denote the group algebra of $G$. The completion of $L_1(G)$ in the minimal regular norm is the group $C^*$-algebra, $C^*(G)$. Let $B(G)$ denote the space of all finite linear combinations of continuous positive definite functions on $G$. Regarded as the dual of $C^*(G)$,
$B(G)$ is a Banach space and even a commutative Banach algebra under pointwise multiplication. Denote by $A(G)$ the closed subspace generated by elements of $B(G)$ with compact support. It is easy to show that $A(G)$ is an ideal in $B(G)$.

For $u \in B(G)$ and $g \in G$, define $\varphi u$ by $\varphi u(h) = u(gh)$. Then $\varphi u \in B(G)$ and $\|\varphi u\| = \|u\|$. Further if $u \in A(G)$, then so does $\varphi u$.

The algebra $A(G)$ behaves in many ways like a group algebra. In general, however, it lacks one important property, the existence of an approximate identity. In [4] Leptin has shown that an approximate identity exists iff $G$ is amenable (see [3] for various alternative characterizations of amenability). We shall use this result in what follows.

3. Centralizers of $A(G)$.

**Theorem 1.** Let $G$ be amenable and let $T$ be a centralizer of $A(G)$. Then there exists a unique element $w \in B(G)$ such that $T = L_w$ (where $L_w$ is defined by $L_wu = uw$ for $u \in A(G)$). Furthermore $\|T\| = \|w\|$.

**Proof.** Let $\{e_\alpha\}$ be an approximate identity for $A(G)$ with $\|e_\alpha\| = 1$. The elements $Te_\alpha$ may be regarded as linear functionals on $C^*(G)$ and by continuity of $T$, the family $\{Te_\alpha\}$ has a $w^*$-accumulation point $w \in B(G)$.

Fix $u \in A(G)$. A straightforward density argument shows that $uw$ is a $w^*$-accumulation point of $\{uTe_\alpha\}$. But then by continuity of $T$,

$$Tu = \lim_{\alpha} T(u e_\alpha) = \lim_{\alpha} u T e_\alpha,$$

This shows that $Tu = uw$, i.e., $T = L_w$.

Since for any $g \in G$ there exists some $u \in A(G)$ with $u(g)$ $\neq 0$, it follows that $w$ is unique. That $\|T\| = \|w\|$ follows from the existence of an approximate identity. Alternately it may be obtained at once from a result of Derighetti [1, Theorem 9].

**Corollary 1.** In the strong operator topology on $A(G)$, $B(G)$ is a closed subset of the algebra of all bounded operators.

**Proof.** For if $\{w_\alpha\}$ is a directed set in $B(G)$ which converges strongly to an operator $S$ on $A(G)$ then for $u, v \in A(G),$

$$S(uv) = \lim_{\alpha} w_\alpha uv = (Su)v$$

so that $S$ is a centralizer. Now apply Theorem 1.

4. Isometric centralizers. We consider now the case when a centralizer $T$ is an isometry i.e. $\|Tu\| = \|u\|$ for all $u \in A(G)$. We prove

**Theorem 2.** Let $G$ be amenable and let $T$ be a centralizer of $A(G)$. Then
$T$ is an isometry iff $T = \lambda L_\kappa$ where $\lambda$ is some scalar of unit modulus and $\kappa$ is a character of $G$.

**Remarks.** 1. It is straightforward to show that an operator $\lambda L_\kappa$ is an isometry so that it suffices to prove the "only if" condition.

2. For some groups $G$, $A(G)$ may not possess any isometric centralizers apart from those of the form $\lambda I$ since there exist amenable groups which have no nontrivial finite dimensional representations.

By Theorem 1, $T$ is of the form $L_w$ for some $w \in B(G)$. The proof of Theorem 2 depends on the following lemmas.

**Lemma 1.** For all $g \in G$, $w(g) \neq 0$.

**Proof.** Firstly note that if $w$ defines an isometry then so does $gw$ for any $g$. For if $u \in A(G)$ then

$$||uw|| = ||w(g)^{-1}u|| = ||w(g)^{-1}u|| = ||g^{-1}u|| = ||u||.$$

Hence it suffices to assume that $w(e) = 0$ and to deduce a contradiction.

Choose $u \in A(G)$ with $u$ positive definite and $||u|| = 1$. Then $wu \in A(G)$ and $wu(e) = 0$. Choose $\varepsilon$ such that $0 < \varepsilon < 1$. By [2, Lemma 4.13], we may choose $v \in A(G)$ with $v$ positive definite, $||v|| = 1$ and $||wv|| < \varepsilon$. But then $\varepsilon > ||wuv|| = ||v|| ||u|| = 1$ which is a contradiction.

**Lemma 2.** $L_w$ maps $A(G)$ onto $A(G)$.

**Proof.** Let $N$ be the range of $L_w$. Since $L_w$ is an isometry, $N$ is a closed ideal in $A(G)$. By a Tauberian theorem [2, Corollary 3.38], it suffices to show that for all $g \in G$ there exists some $v \in N$ with $v(g) \neq 0$. Choose $u \in A(G)$ with $u(g) \neq 0$ and let $v = uw$. The result now follows from Lemma 1.

**Proof of Theorem 2.** From Lemma 2, $L_w^{-1}$ exists as a bounded operator on $A(G)$. If $u, v \in A(G)$ with $u = u_1w$ then $L_w^{-1}(uv) = L_w^{-1}(L_w(u)v) = u_1v = (L_w^{-1}u)v$ so that $L_w^{-1}$ is a centralizer. From this it follows that $w^{-1} \in B(G)$ with $||w^{-1}|| = 1$.

Now $||w||_\infty = \sup_{g \in G} ||w(g)|| \leq ||w|| = 1$ and similarly $||w^{-1}||_\infty \leq 1$. Hence $||w(g)|| = 1$ for all $g \in G$.

We can write $w(g) = (U_g\xi, \eta)$ where $g \to U_g$ is a unitary representation of $G$ on some Hilbert space $H$ with cyclic vector $\xi$ and with $||w|| = ||\xi|| ||\eta||$. For all $g \in G$,

$$1 = ||w(g)|| = ||(U_g\xi, \eta)|| \leq ||\xi|| ||\eta|| = 1$$

so that by the Cauchy-Schwarz inequality, $H$ is one dimensional and there exist scalars $\lambda_g$ of unit modulus such that $U_g\xi = \lambda_g \eta$. Let $\lambda = \lambda_e^{-1}$. Then if $\kappa(g) = \lambda^{-1}\lambda_g$, we have that $\kappa$ is a character of $G$ and $w = \lambda \kappa$. This proves the theorem.
References