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SUBMODULES OF $C(X) \times \cdots \times C(X)$

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ABSTRACT. Let $C(X)$ be the ring of continuous real-valued functions on a compact Hausdorff space X with the sup norm topology. In the present paper, the closed submodules of the $C(X)$-module $C(X) \times \cdots \times C(X)$ are characterized and a necessary and sufficient condition for a submodule of $C(X) \times \cdots \times C(X)$ to be closed and finitely generated is established.

1. **Introduction.** Let $C(X)$ be the ring of continuous real-valued functions on a compact Hausdorff space X with the sup norm topology where $\|f\| = \sup\{|f(x)| : x \in X\}$ for $f \in C(X)$. Let $[C(X)]^p$ denote the cartesian product of $C(X)$ with itself p-times with the product topology which can be defined by the norm $\|f\| = \max\{\|f_1\|, \cdots, \|f_p\|\}$ for $f = (f_1, \cdots, f_p) \in [C(X)]^p$.

$[C(X)]^p$ is a $C(X)$-module. We are here concerned with two questions:

(1) Which submodules of $[C(X)]^p$ are closed? Equivalently, what is the closure of a submodule of $[C(X)]^p$?

(2) Which submodules of $[C(X)]^p$ are closed and finitely generated?

These questions can be viewed as both generalizations and specializations of previous results.

Questions (1) and (2) are natural generalizations of questions about ideals in $C(X)$ which arise when $p=1$. It is well known that an ideal in $C(X)$ is closed if and only if the ideal contains every function in $C(X)$ which vanishes on the zero set of the ideal [**4**, Theorem 8, p. 53]. And an ideal in $C(X)$ is closed and finitely generated if and only if the zero set of the ideal is an open set in X [**2**, Theorem 2.1]. However, it is easy to see that the notion of the zero set of a submodule is inappropriate for answering (1) and (2). For example, the proper closed submodule $M = \{(f, f) : f \in C(X)\}$ of $C(X) \times C(X)$ has no zeros, i.e., for each i, $1 \leqq i \leqq 2$, there is no point $x_i \in X$ such that $f_i(x_i) = 0$ for all $(f_1, f_2) \in M$.

Also, questions (1) and (2) are special cases of deeper questions about submodules of differentiable functions. Malgrange [**1**, Theorem 1.3, p. 22]

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has established the more difficult characterization of the closed submodules of differentiable functions by generalizing Whitney's theorem from ideals to submodules. The answer to (1) in Theorem 2.1 is a special case of Malgrange's result, although the proof in this case is simplified by the absence of differentiation. Theorem 3.1, the answer to (2), indicates the general nature of the results to be expected in the study of closed finitely generated submodules of differentiable functions. Interest in closed finitely generated submodules of differentiable functions stems primarily from their connection by duality with systems of problems of division for distributions (see [**3**, Theorem 3.1]).

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2. **Closed submodules of** $[C(X)]^p$. Consider a submodule M of $[C(X)]^p$. For each $x \in X$, let $E(x)=\{f(x): f \in M\}$. We say that $f \in [C(X)]^p$ is *pointwise in* M if for each $x \in X$, $f(x) \in E(x)$. The set $\hat{M}$ of all $f \in [C(X)]^p$ which are pointwise in M is a closed submodule of $[C(X)]^p$ containing M. And, in fact, $\hat{M}$ is the closure of M.

THEOREM 2.1. *Suppose M is a submodule of* $[C(X)]^p$. *Then* $\mathrm{cl}(M)=\hat{M}$.

PROOF. Clearly $\mathrm{cl}(M) \subset \hat{M}$. Suppose $f \in \hat{M}$ and consider $\varepsilon>0$. We show that there exists $g \in M$ such that $\|f-g\|<\varepsilon$. Since f is pointwise in M, for each $x \in X$ there exists $g_x \in M$ such that $f(x)=g_x(x)$. Adopting the notation $|h(x)|=\max\{|h_1(x)|, \cdots, |h_p(x)|\}$ for $h=(h_1, \cdots, h_p) \in [C(X)]^p$, we find that for each $x \in X$ there exists an open neighborhood N_x of x such that $|f(y)-g_x(y)|<\varepsilon$ for all $y \in N_x$. Since X is compact, there exist finitely many of these open neighborhoods, say $\{N_i\}_{1 \leqq i \leqq n}$ associated with $\{g_i\}_{1 \leqq i \leqq n}$, which cover X. Let $\{\alpha_i\}_{1 \leqq i \leqq n}$ be a partition of unity subordinate to the covering $\{N_i\}_{1 \leqq i \leqq n}$ and let $g=\sum_{i=1}^n \alpha_i g_i$. Then $g \in M$ and, moreover, $\|f-g\|<\varepsilon$. For if $x \in X$, then

$$|f(x) - g(x)| \leqq \sum_{i=1}^{n} \alpha_i(x)\, |f(x) - g_i(x)| < \sum_{i=1}^{n} \alpha_i(x)\varepsilon = \varepsilon$$

since if $\alpha_i(x) \neq 0$, then $x \in N_i$ so $|f(x)-g_i(x)|<\varepsilon$. Hence $\|f-g\|<\varepsilon$ and therefore $f \in \mathrm{cl}(M)$. This completes the proof of Theorem 2.1.

COROLLARY 2.1. *M is a closed submodule of* $[C(X)]^p$ *if and only if $f \in M$ for all $f \in [C(X)]^p$ which are pointwise in M.*

3. **Closed finitely generated submodules of** $[C(X)]^p$. Consider a submodule M of $[C(X)]^p$. For $0 \leqq k \leqq p$, let $R_k=\{x \in X: \dim(E(x))=k\}$. In order that M be closed and finitely generated, it is necessary and sufficient that each R_y be open.

THEOREM 3.1. *A submodule M of $[C(X)]^p$ is closed and finitely generated if and only if R_k is an open set in X for $0\leqq k\leqq p$.*

PROOF. *Necessity*. Suppose that M is a closed finitely generated submodule of $[C(X)]^p$, say $M=\{g_1\boldsymbol{f}_1+\cdots+g_q\boldsymbol{f}_q:g_1,\cdots,g_q\in C(X)\}$ where $\boldsymbol{f}_j=(f_{1j},\cdots,f_{pj})\in[C(X)]^p$ for $1\leqq j\leqq q$. Let F denote the $p\times q$ matrix $(f_{ij})_{1\leqq i\leqq p,1\leqq j\leqq q}$. Then $F:[C(X)]^q\to[C(X)]^p$ by matrix multiplication with $\operatorname{im}(F)=M$ and $R_k=\{x\in X:\operatorname{rank}(F(x))=k\}$ for $0\leqq k\leqq p$. For $0\leqq k\leqq p$, let $S_k=\{x\in X:\operatorname{rank}(F(x))\geqq k\}$. Clearly each S_k is an open set in X and $S_k\subset S_{k-1}$ for $1\leqq k\leqq p$.

To show that each R_k is open, it suffices to show that S_k is closed for $1\leqq k\leqq p$ since $R_k=S_k-S_{k+1}$ for $0\leqq k<p$. And to show that S_k is closed in X for $1\leqq k\leqq p$, we prove by induction for $1\leqq k\leqq p$ the property $P(k)$ which states that there exists a constant $C_k>0$ such that the maximum of the absolute values of the determinants of the $k\times k$ submatrices of $F(x)$ is greater than or equal to C_k for all $x\in S_k$. Clearly this implies that S_k is a closed set in X for $1\leqq k\leqq p$.

Since F is a continuous linear mapping of the Banach space $[C(X)]^q$ onto the Banach space $M=\operatorname{im}(F)$, F is an open mapping. Hence there exists a constant $C>0$ such that if $\boldsymbol{h}\in\operatorname{im}(F)$, then there exists $\boldsymbol{g}\in[C(X)]^q$ with $F(\boldsymbol{g})=\boldsymbol{h}$ and $\|\boldsymbol{g}\|\leqq C\|\boldsymbol{h}\|$.

To prove $P(1)$, consider $x\in S_1$. For simplicity suppose $|f_{11}(x)|=\max\{|f_{ij}(x)|:1\leqq i\leqq p,\ 1\leqq j\leqq q\}$. Then $f_{11}(x)\neq 0$ since $x\in S_1$ and therefore f_{11} is nonzero in an open neighborhood N of x. By Urysohn's lemma, there exists $h\in C(X)$ with $h(x)=1$, $\|h\|=1$, and $\operatorname{supp}(h)\subset N$. Then $\boldsymbol{h}=(f_{11}h/f,\cdots,f_{p1}h/f)\in\operatorname{im}(F)$ where $f=|f_{11}|+\cdots+|f_{p1}|$, and $\|\boldsymbol{h}\|\leqq 1$. Hence there exists $\boldsymbol{g}=(g_1,\cdots,g_q)\in[C(X)]^q$ with $F(\boldsymbol{g})=\boldsymbol{h}$ and $\|\boldsymbol{g}\|\leqq C\|\boldsymbol{h}\|\leqq C$. Thus

$$f_{11}g_1+\cdots+f_{1q}g_q=f_{11}h/f.$$

Now $|f(x)|\leqq p|f_{11}(x)|$ and $h(x)=1$. Hence

$$|f_{11}(x)g_1(x)|+\cdots+|f_{1q}(x)g_q(x)|\geqq 1/p$$

so for some j, $1\leqq j\leqq q$, we have $|f_{1j}(x)g_j(x)|\geqq 1/pq$. Thus

$$1/|f_{1j}(x)|\leqq pq\,|g_j(x)|\leqq pq\,\|\boldsymbol{g}\|\leqq pqC.$$

Therefore, for each $x\in S_1$, $|f_{ij}(x)|\geqq 1/pqC$ for some i and j, $1\leqq i\leqq p$, $1\leqq j\leqq q$, which proves $P(1)$.

To prove that $P(k)$ implies $P(k+1)$ where $1\leqq k<p$, assume $P(k)$ and consider $x\in S_{k+1}$. Then $x\in S_k$ so the absolute value of the determinant of some $k\times k$ submatrix of $F(x)$ is greater than or equal to C_k, say for simplicity $|\det(f_{ij}(x))_{1\leqq i,j\leqq k}|\geqq C_k$. A simple linear dependence argument establishes that some $(k+1)\times(k+1)$ submatrix of $F(x)$ containing $(f_{ij}(x))_{1\leqq i,j\leqq k}$ is nonsingular since $x\in S_{k+1}$.

Assume for simplicity that $|\det(f_{ij}(x))_{1\leqq i,j\leqq k+1}|$ is the maximum of the absolute values of the determinants of the $(k+1)\times(k+1)$ submatrices of $F(x)$ containing $(f_{ij}(x))_{1\leqq i,j\leqq k}$. Then $(f_{ij})_{1\leqq i,j\leqq k+1}$ is nonsingular in an open neighborhood N of x. By Urysohn's lemma, there exists $h\in C(X)$ with $h(x)=1$, $\|h\|=1$ and $\operatorname{supp}(h)\subset N$.

For $1\leqq i\leqq p$, let

$$f_i = \det\begin{pmatrix} f_{11} & \cdots & f_{1(k+1)} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ f_{k1} & \cdots & f_{k(k+1)} \\ f_{i1} & \cdots & f_{i(k+1)} \end{pmatrix}.$$

And for $1\leqq i\leqq p$, let $h_i=f_ih/f$ where $f=|f_1|+\cdots+|f_p|$. Then $\boldsymbol{h}=(h_1,\cdots,h_p)\in\operatorname{im}(F)$ and $\|\boldsymbol{h}\|\leqq 1$. Hence there exists $\boldsymbol{g}=(g_1,\cdots,g_q)\in[C(X)]^q$ with $F(\boldsymbol{g})=\boldsymbol{h}$ and $\|\boldsymbol{g}\|\leqq C\|\boldsymbol{h}\|\leqq C$. Thus

$$\begin{aligned} f_{11}g_1 + \cdots + f_{1q}g_q &= h_1 = 0 \\ \cdot \qquad\qquad \cdot \quad &\quad\; \cdot \quad\; \cdot \\ \cdot \qquad\qquad \cdot \quad &\quad\; \cdot \quad\; \cdot \\ \cdot \qquad\qquad \cdot \quad &\quad\; \cdot \quad\; \cdot \\ f_{k1}g_1 + \cdots + f_{kq}g_q &= h_k = 0 \\ f_{(k+1)1}g_1 + \cdots + f_{(k+1)q}g_q &= h_{k+1} = f_{k+1}h/f. \end{aligned}$$

In this system of equations, multiply the ith row by the cofactor of $f_{i(k+1)}$ in $(f_{ij})_{1\leqq i,j\leqq k+1}$. Adding the equations thus obtained, we find that the coefficient of g_j for $1\leqq j\leqq q$ is

$$d_j = \det\begin{pmatrix} f_{11} & \cdots & f_{1k} & f_{1j} \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ f_{(k+1)1} & \cdots & f_{(k+1)k} & f_{(k+1)j} \end{pmatrix}.$$

Since $d_j=0$ for $1\leqq j\leqq k$, we have

$$d_{k+1}g_{k+1} + \cdots + d_qg_q = \det(f_{ij})_{1\leqq i,j\leqq k}f_{k+1}h/f.$$

Now $|\det(f_{ij}(x))_{1\leqq i,j\leqq k}|\geqq C_k$, $|f(x)|\leqq p\,|f_{k+1}(x)|$, and $h(x)=1$. Hence

$$|d_{k+1}(x)g_{k+1}(x)| + \cdots + |d_q(x)g_q(x)| \geqq C_k/p$$

so for some j, $k+1\leqq j\leqq q$, we have $|d_j(x)g_j(x)|\geqq C_k/pq$. Thus

$$1/|d_j(x)| \leqq (pq/C_k)|\,g_j(x)| \leqq (pq/C_k)\;\|\boldsymbol{g}\| \leqq pqC/C_k.$$

Since $|d_j(x)|$ for $k+1\leqq j\leqq q$ is the absolute value of the determinant of a $(k+1)\times(k+1)$ submatrix of $F(x)$, $P(k+1)$ is established.

Sufficiency. Suppose that for $0\leqq k\leqq p$, R_k is open in X. We prove that M is closed and finitely generated by showing that for $0\leqq k\leqq p$, there exists a finite subet of M which generates $\hat{M}$ in R_k. In R_0, this is clearly the case.

Consider R_k where $1\leqq k\leqq p$. For each $x\in R_k$, there exists $\boldsymbol{f}_1,\cdots,\boldsymbol{f}_k\in M$ where $\boldsymbol{f}_j=(f_{1j},\cdots,f_{pj})$ for $1\leqq j\leqq k$ such that some $k\times k$ submatrix of $(f_{ij})_{1\leqq i\leqq p,1\leqq j\leqq k}$ is nonsingular in a neighborhood of x. Since R_k is compact, we obtain a finite open covering of R_k consisting of subsets of R_k in this fashion.

Consider a member of this finite open covering, say N associated with $\boldsymbol{f}_1,\cdots,\boldsymbol{f}_k\in M$ where for simplicity we suppose that $(f_{ij})_{1\leqq i,j\leqq k}$ is nonsingular in N. Then $\boldsymbol{f}_1,\cdots,\boldsymbol{f}_k$ generate $\hat{M}$ in N. For suppose $\boldsymbol{h}=(h_1,\cdots,h_p)\in\hat{M}$. Then the system of equations

$$\begin{pmatrix} f_{11}g_1+\cdots+f_{1k}g_k=h_1\\ \cdot \qquad\qquad \cdot \quad \cdot\\ \cdot \qquad\qquad \cdot \quad \cdot\\ \cdot \qquad\qquad \cdot \quad \cdot\\ f_{k1}g_1+\cdots+f_{kk}g_k=h_k\end{pmatrix}$$

can be uniquely solved in N for functions $g_1,\cdots,g_k\in C(N)$. We claim that $g_1\boldsymbol{f}_1+\cdots+g_k\boldsymbol{f}_k=\boldsymbol{h}$ in N since for $1\leqq i\leqq p$, we have

$$h_i-\left(\sum_{j=1}^{k}g_jf_{ij}\right)=\det\begin{pmatrix} f_{11} & \cdots & f_{1k} & h_1\\ \cdot & & \cdot & \cdot\\ \cdot & & \cdot & \cdot\\ \cdot & & \cdot & \cdot\\ f_{k1} & \cdots & f_{kk} & h_k\\ f_{i1} & \cdots & f_{ik} & h_i\end{pmatrix}\Big/\det(f_{ij})_{1\leqq i,j\leqq k}$$

in N. Now for all $x\in N$, $\boldsymbol{f}_1(x),\cdots,\boldsymbol{f}_k(x)$ form a basis for $E(x)$ and hence $\boldsymbol{h}(x)\in E(x)$ is linearly dependent on $\boldsymbol{f}_1(x),\cdots,\boldsymbol{f}_k(x)$ for all $x\in N$. Thus

$$\det\begin{matrix} f_{11}(x) & \cdots & f_{1k}(x) & h_1(x)\\ \cdot & & \cdot & \cdot\\ \cdot & & \cdot & \cdot\\ \cdot & & \cdot & \cdot\\ f_{k1}(x) & \cdots & f_{kk}(x) & h_k(x)\\ f_{i1}(x) & \cdots & f_{ik}(x) & h_i(x)\end{matrix}=0$$

for all $x\in N$ and $1\leqq i\leqq p$ and therefore $g_1\boldsymbol{f}_1+\cdots+g_k\boldsymbol{f}_k=\boldsymbol{h}$ in N.

Suppose the finite open covering of R_k is $\{N_i\}_{1\leqq i\leqq n}$ associated with $\{f_1^i, \cdots, f_k^i\}_{1\leqq i\leqq n}$. Then for $1\leqq i\leqq n$, the functions $f_1^i, \cdots, f_k^i \in M$ generate $\hat{M}$ in R_k. For consider $h\in\hat{M}$. Then for $1\leqq i\leqq n$, there exists $g_1^i, \cdots, g_k^i \in C(N_i)$ with $g_1^i f_1^i + \cdots + g_k^i f_k^i = h$ in N_i. Let $\{\alpha_i\}_{1\leqq i\leqq n}$ be a partition of unity subordinate to the covering $\{N_i\}_{1\leqq i\leqq n}$ with $\mathrm{supp}(\alpha_i)\subset N_i$ for $1\leqq i\leqq n$. Extending each g_j^i from $\mathrm{supp}(\alpha_i)$ to $g_j^{i\prime}\in C(R_k)$ by the Tietze extension theorem, we have

$$\sum_{i=1}^{n} \alpha_i g_1^{i\prime} f_1^i + \cdots + \alpha_i g_k^{i\prime} f_k^i = \sum_{i=1}^{n} \alpha_i h = h$$

in R_k.

COROLLARY 3.1. *If X is connected, a submodule M of $[C(X)]^p$ is closed and finitely generated if and only if* $\dim(E(x))$ *is constant for $x\in X$.*

COROLLARY 3.2. *An ideal I in $C(X)$ is closed and finitely generated if and only if $Z(I)=\{x\in X: f(x)=0$ for all $f\in I\}$ is an open set in X.*

PROOF. Immediate from Theorem 3.1 since $R_0=Z(I)$.

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