PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 32, Number 2, April 1972

MAXIMAL INDEPENDENT COLLECTIONS OF CLOSED SETS

HARVY LEE BAKER, JR.

ABSTRACT. A theorem is proved which implies that if X is a separable metric space then there exists a *countable* maximal independent subset of the lattice of closed subsets of X. In the case where X has no isolated points this independent set is nontrivial in the sense that X does not belong to it and it contains no singletons. Furthermore, if X is a compact metric continuum such that $\bigcup \{O \mid O$ is an open subset of X and O is homeomorphic to E^n for some $n > 1\}$ is dense in X then there exists a countable maximal such collection whose elements are *connected*. This complements previous work by the author which characterized continua for which there are such collections of a specialized nature.

1. An independent subset of a partially ordered set is one such that no two elements of it are comparable. See [5]. An *amonotonic* collection of sets is one which is independent relative to set inclusion. A *complete amonotonic decomposition* of a connected T_1 space T is a maximal amonotonic collection of closed connected subsets of T which is nontrivial in the sense that it contains at least two elements and at least one element of it is not a singleton.

No compact metric continuum has a finite complete amonotonic decomposition (Theorem 1.1 of [3]), whereas many have countable such decompositions and some have only uncountable ones (arcs, indecomposable continua, etc.). Theorem 2.3 is the main object in the present paper and describes maximal independent subsets of the lattice of closed subsets of many T_3 spaces (all those having "pseudodevelopments"). The cardinality of the maximal independent subset obtained by use of this theorem is often less than the cardinality of the topology of the space, and in particular it follows that if X is a separable metric nondegenerate space then there is a *countable* maximal independent subset of the lattice of closed subsets of X which is nontrivial in the sense that it does not contain X, and if the set of isolated points of X is not dense in X then the only singletons belonging to it are isolated points. Theorem 2.3 also implies that if T

Received by the editors January 20, 1970.
AMS 1969 *subject classifications*. Primary 5440; Secondary 5455, 5425.
Key words and phrases. Independent subset of a partially ordered set, amonotonic collection of sets, complete amonotonic decomposition of a continuum, pseudodevelopment of a space.

is a compact metric continuum and

$$\bigcup \{O \mid O \text{ is open in } T \text{ and } O \cong E^n \text{ for some } n \geqq 2\}$$

is dense in T then there exists a countable complete amonotonic decomposition of T. An even larger class of continua which have countable complete amonotonic decompositions is given in Theorem 2.5. Even this theorem, however, falls short of characterizing those continua which do have such decompositions. For example in **[3]** the following statements concerning a compact metric continuum M are shown to be equivalent: (A) There is a complete amonotonic decomposition G of M such that if g_1 and g_2 are distinct elements of G then $M = g_1 \cup g_2$, and $M - g_1$ is connected. (B) There is a monotone continuous mapping F from M onto a simple closed curve J such that if H is a proper closed subset of M then $F(H) \neq J$. It is impossible to obtain a countable complete amonotonic decomposition of such an M using Theorem 2.5 (note that statement (A) implies that G is countable). Finite graphs with no end points, and continua related to them the way M is related to S^1 in statement (B), also have countable complete amonotonic decompositions (see **[4]** and **[2]**), but no such decomposition can be obtained by application of Theorem 2.5.

It is of interest to note that the complete amonotonic decompositions obtained in **[4]** for finite graphs with no end points, and the decomposition G obtained in statement (A), are *not* maximal independent subsets of the lattice of closed subsets of the respective spaces, but that the decompositions obtained in Theorem 2.5 and Theorem 2.6 (for "near manifolds" T) are indeed maximal in this lattice.

2. The following definitions introduce terminology used in the statement of Theorem 2.3 and throughout the proof of it.

DEFINITION 2.1. A pseudodevelopment of the T_3 space (X, T) is a sequence $G = G_1, G_2, \cdots$ of nonempty collections of disjoint open sets such that (a) for each n, G_{n+1} refines G_n and $\mathrm{Cl}\,(G_n^*) = X$ (the star is the union), and (b) if O is a nonempty open set then there is an n such that some element of G_n is a subset of O.

DEFINITION 2.2. If $G = G_1, G_2, \cdots$ is a pseudodevelopment of the T_3 space (X, T) then $W(G)$ will denote the collection to which X belongs if and only if there exists a positive integer j and j distinct elements $X_1, X_2, \cdots, X_j$ of G_j such that (a) $X = \text{Interior}\,(\bar{X}_1 \cup \bar{X}_2 \cup \cdots \cup \bar{X}_j)$, and (b) if i is an integer such that $1 \leqq i < j$ then $X_1 \cup X_2 \cup \cdots \cup X_j$ is not a subset of the union of i or fewer elements of G_i.

THEOREM 2.3. *If $G = G_1, G_2, \cdots$ is a pseudodevelopment of the T_3 space (X, T) and for each n, each element of G_n contains at least two elements of G_{n+1}, then $W(G)$ is a maximal independent subset of T.*

PROOF. If for each n we let W_n denote the sets of elements of $W(G)$ which are determined by n distinct elements of G_n, then a routine argument shows that W_1 is an amonotonic collection and that if $W_1 \cup W_2 \cup \cdots \cup W_i$ is an amonotonic collection then so is $W_1 \cup W_2 \cup \cdots \cup W_{i+1}$. Hence $W(G)$ is an amonotonic collection.

It must be shown now that every open set either contains or is a subset of an element of $W(G)$. Suppose D is an open set. Suppose there exists an integer i such that D intersects at most i elements of G_i. We will show that in this case D is a subset of some set of $W(G)$. Let $O_1, O_2, \cdots, O_n$ denote the distinct elements of G_i which intersect D. We note that since, for each i, the elements of G_i are disjoint and $\mathrm{Cl}\,(G_i^*) = X$, the set $I =$ Interior $(\bar{O}_1 \cup \bar{O}_2 \cup \cdots \cup \bar{O}_n)$ must contain D. Hence if we can establish that I is subset of a set of $W(G)$ we are through with this portion of the argument.

That this is so follows from the seemingly more general proposition: If $D_1, D_2, \cdots, D_m$ are distinct elements of G_h where $m \leqq h$ then Interior $(\bar{D}_1 \cup \bar{D}_2 \cup \cdots \cup \bar{D}_m)$ is a subset of a set of $W(G)$. We prove this proposition by letting e denote the *least* positive integer such that $D_1 \cup D_2 \cup \cdots \cup D_m$ is a subset of e or fewer elements of G_e. We know however that G_e contains at least 2^{e-1}, hence at least e, elements. Thus we can find e distinct elements $h_1, h_2, \ldots, h_e$ of G_e such that $D_1 \cup D_2 \cup \cdots \cup D_m \subset h_1 \cup h_2 \cup \cdots \cup h_e$. Now $h_1 \cup h_2 \cup \cdots \cup h_e$ satisfies (a) of Definition 2.2, and if (b) is not satisfied then there is an l such that $1 \leqq l < e$ and $h_1 \cup h_2 \cup \cdots \cup h_e$ is a subset of the union of l distinct elements of G_l. But in this case e is no longer the *least* positive integer such that $D_1 \cup D_2 \cup \cdots \cup D_m$ is a subset of e or fewer elements of G_e, a contradiction. Hence $\mathrm{Int}\,(\bar{h}_1 \cup \bar{h}_2 \cup \cdots \cup \bar{h}_e) \in W(G)$, and since $D_1 \cup D_2 \cup \cdots \cup D_m \subset h_1 \cup h_2 \cup \cdots \cup h_e$ implies

$$\mathrm{Int}\,(\bar{D}_1 \cup \bar{D}_2 \cup \cdots \cup \bar{D}_m) \subset \mathrm{Int}\,(\bar{h}_1 \cup \bar{h}_2 \cup \cdots \cup \bar{h}_e),$$

we are through.

Suppose now that for each i, D intersects at least $i + 1$ elements of G_i. We will show that D contains a set of $W(G)$. There exist an integer k and two distinct elements X_1 and X_2 of G_k such that $\bar{X}_1 \cup \bar{X}_2 \subset D$ and X_1 and X_2 lie in the same element of G_{k-1}. Let h_1^1 denote the element of G_1 which contains $\bar{X}_1 \cup \bar{X}_2$ and let h_1^2 be an element of G_1 distinct from h_1^1 which also intersects D. Now let h_2^1 denote the element of G_2 which lies in h_1^1 and contains $\bar{X}_1 \cup \bar{X}_2$, let h_2^2 denote an element of G_2 which lies in h_1^2 and intersects D, and let h_2^3 denote any element of G_2 which intersects D and is distinct from h_2^1 and h_2^2. We proceed by choosing four distinct elements h_3^1, h_3^2, h_3^3, h_3^4 of G_3 such that each intersects D and such that $h_3^1 \subset h_2^1$, $h_3^2 \subset h_2^2$, $h_3^3 \subset h_2^3$ and $\bar{X}_1 \cup \bar{X}_2 \subset h_3^1$. We continue in this manner until we finally obtain k distinct elements $h_{k-1}^1, h_{k-1}^2, \cdots, h_{k-1}^k$ of G_{k-1} such that each intersects D, $\bar{X}_1 \cup \bar{X}_2 \subset h_{k-1}^1$ and $h_{k-1}^a \subset h_{k-2}^a$ for $a = 1, 2, \cdots, k-1$. Notice that the entire set of h_i^j's has been constructed in such a way that $h_a^b \subset h_c^b$ if $a \geqq c$ and $b \leqq c + 1$. A consequence of this (one that will be used later) is that at the $(n + 1)$th stage of the construction, $h_{n+1}^1, h_{n+1}^2, \cdots, h_{n+1}^{n+2}$ are chosen in such a way that the union of all of them does not lie in the union of n elements of G_n. This implies directly that if $m < n$ then $\bigcup h_{n+1}^i$ is not a subset of the union of m or fewer elements of G_m.

Since (X, T) is T_3 and G is a pseudodevelopment of (X, T), there is an integer $d > k$ such that for every a, where $2 \leqq a \leqq k$, $h_{k-1}^a \cap D$ contains the closure of an element of G_d. Let g_a, for $a = 2, 3, \cdots, k$, be an element of G_d whose closure lies in $D \cap h_{k-1}^a$. Choose now distinct elements $g_1^1, g_1^2, \cdots, g_1^{d+1-k}$ of G_d such that $g_1^1 \cup g_1^2 \cup \cdots \cup g_1^{d+1-k} \subset X_1 \cup X_2$ and such that if $k + 1 \leqq i \leqq d$ then no two g_1^n's lie in the same element of G_i. That this can be done is evident from the facts that X_1 and X_2 contain distinct elements x_{11}, x_{12}, and x_{21}, x_{22} of G_{k+1} respectively, that x_{11}, x_{12}, x_{21}, x_{22} contain distinct elements x_{111}, x_{112}, etc. of G_{k+2}, and so forth. We could if we wished choose $2^{d-k} g_1^n$'s meeting the above requirements.

Let $I = g_1^1 \cup g_1^2 \cup \cdots \cup g_1^{d+1-k} \cup g_2 \cup g_3 \cup \cdots \cup g_k$. The closure of each of the g's is in D, so Interior $\bar{I} \subset D$. It will be shown now that Interior $\bar{I} \in W(G)$. Suppose $1 \leqq e < d$ and that I is a subset of the union of e distinct elements of G_e. If $k - 1 \leqq e < d$, then since the h_{k-1}^i's are distinct and $g_a \subset h_{k-1}^a$ for $a \geqq 2$, it follows that the set

$$g_1^1 \cup g_1^2 \cup \cdots \cup g_1^{d+1-k}$$

lies in the union of $e - (k - 1) = e + 1 - k$ elements of G_e. But $e < d$ and, therefore, at least two of the elements $g_1^1, g_1^2, \cdots, g_1^{d+1-k}$ lie in the same element of G_e. This contradicts one of the defining properties of the g_1^i's as given in the last paragraph. Hence, $1 \leqq e < k - 1$. In this case, since $\{h_{k-1}^1, h_{k-1}^2, \cdots, h_{k-1}^k\}$ properly covers I, it follows that

$$h_{k-1}^1 \cup h_{k-1}^2 \cup \cdots \cup h_{k-1}^k$$

is a subset of the union of e elements of G_e where $e < k - 1$. But this contradicts the concluding three statements of the third paragraph of this proof (one of which in effect says that this is one of the properties inherited by the h_{k-1}^i's as a result of the manner in which they were selected). Therefore Interior $\bar{I} \subset D$ and Interior $\bar{I} \in W(G)$, completing the argument.

It is a consequence of the argument above that if D is not a subset of any element of $W(G)$ then it contains infinitely many elements of $W(G)$. This follows since either d or k can be replaced by any larger integer.

Corresponding to every maximal independent subset W of the topology T on X is a maximal independent subset $W' = \{X - w \mid w \in W\}$ of the lattice of closed subsets of X. This correspondence gives us the following result.

THEOREM 2.4. *If M is a separable metric space, then there exists a countable maximal independent subset C of the lattice of closed subsets of M. In the case where M contains no isolated points (i.e. every point of M is a limit point of M) C is nontrivial in the sense that $M \notin C$ and C contains no singletons.*

PROOF. If M is a separable metric space with no isolated points then it is easy to construct pseudodevelopments of M which satisfy the hypothesis of Theorem 2.3. For example: For every open set O of M let $G(O)$ denote a maximal collection of disjoint open subsets of O whose diameters are less than one half the diameter of O. The sequence defined by $G_1 = G(M)$, $G_{n+1} = \bigcup \{G(x) \mid x \in G_n\}$ is a pseudodevelopment of M satisfying the conditions of Theorem 2.3. Hence there is as a consequence of the remark preceding the statement of this theorem, a maximal independent subset of the lattice of closed subsets of M which has the same cardinality as $W(G)$, where $G = G_1, G_2, \cdots$. But $W(G)$ is countable since it is a subset of the set of all finite subsets of $\bigcup G_n$, and G_i is countable for each i since M is separable and the elements of G_i are disjoint open sets.

If M has an isolated point P then $\{M - P, \{P\}\}$ is a maximal independent collection of closed subsets of M. If $M_2 = M - \bar{M}_1$ is nonempty, where M_1 is the set of all isolated points of M, then there is a nontrivial countable maximal independent subset of the lattice of closed subsets of M of the form $W_1 \cup \{\{P\} \mid P$ is an isolated point of $M\}$, where W_1 is a maximal independent collection of closed subsets of M_2 obtained by application of the results of the first part of this argument.

THEOREM 2.5. *If M is a separable metric space, $G = G_1, G_2, \cdots$ is a pseudodevelopment of M satisfying the hypothesis of Theorem* 2.3, *and $M - A^*$ is connected for every subcollection A of $\bigcup G_n$, then $W' = \{M - x \mid x \in W(G)\}$ is a countable complete amonotonic decomposition of M. Furthermore, W' is also a maximal independent subset of the lattice of closed subsets of M.*

THEOREM 2.6. *If T is a compact metric continuum and $\bigcup_{n>1} T_n$ is dense in T, where T_n is the union of all open subsets of T which are homeomorphic to E^n, then there exists a countable complete amonotonic decomposition of T which is also a maximal independent collection of closed subsets of T.*

PROOF. The space $\bigcup_{n>1} T_n$ is a separable metric space and has as a basis the collection of all open cells g such that (a) $\bar{g} \subset T_n$ for some n, (b) the diameter of g is less than the distance from g to $T - \bigcup_{n>1} T_n$.

Open cell here is an open subset of T which is homeomorphic to E^n for some n and whose closure is a closed cell. From this basis (or any other basis) a pseudodevelopment $G = G_1, G_2, \cdots$ can be constructed such that the boundaries of the elements of $\bigcup G_n$ are disjoint. Now $W(G)$ is a maximal independent subset of the topology of $\bigcup_{n>1} T_n$ and, by condition (b) and the fact that $\mathrm{Cl}\,(\bigcup_{n>1} T_n) = T$, we know that every open set of T which intersects $T - \bigcup_{n>1} T_n$ contains an element of G_1 and hence an element of $W(G)$. Here we should note that $\mathrm{Int}\,\bar{g} = g$ for open cells g.

Thus $W(G)$ is a maximal independent subset of the topology on T and the set of complements of the elements of $W(G)$ is a countable maximal independent subset of the lattice of closed subsets of T. The elements of $W(G)$ have connected complements since the components of $\bigcup_{n>1} T_n$ are connected manifolds of dimension greater than one, no such manifold is disconnected by the union of a finite number of open cells whose boundaries are disjoint, and no element of $\bigcup G_n$ has a boundary which intersects $T - \bigcup_{n>1} T_n$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NEBRASKA, LINCOLN, NEBRASKA 68508