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RECURRENCE AND ALMOST PERIODICITY IN A GENERATIVE TRANSFORMATION GROUP

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ABSTRACT. A point p in a transformation group (X, T, Π) is recurrent if for every neighborhood U of p there is an extensive set $E \subset T$ such that $pE \subset U$. The point p is almost periodic if there is a syndetic set $A \subset T$ such that $pA \subset U$. This paper proves that a recurrent point in a locally compact, generative transformation group with an equicontinuous neighborhood must also be almost periodic.

The purpose of this paper is to prove that, under certain conditions, a recurrent point must also be almost periodic. The definitions are as in the book [**1**]. All spaces are assumed to be Hausdorff.

THEOREM. *Let (X, T, Π) be a transformation group, where X is a locally compact uniform space and T is generative and equicontinuous in a neighborhood of a point p. If p is recurrent then p is almost periodic.*

The proof of the theorem will make use of the following remarks, which are known for the most part.

REMARK 1. T is topologically isomorphic to $C \times I^l \times R^m$, where I is the integers with the discrete topology, R is the reals with the usual topology and C is a compact abelian group.

REMARK 2. If E is an extensive subset of T, then for any specified coordinates in the representation $C \times I^l \times R^m$ a sequence on E can be found which increases without bound in these coordinates and decreases without bound in the others.

PROOF. This follows directly from the definition of extensive.

REMARK 3. A subset A of T is syndetic if and only if there exists a compact subset K of T such that $tK \cap A \neq \varnothing$ for every t in T.

PROOF OF THE THEOREM. Let U be an open neighborhood of p and let $A = \{t \mid t \in T, pt \in U\}$. Then A is extensive and we wish to show that A is syndetic. Suppose that this is not the case.

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Let U^* be an open neighborhood of p such that T is equicontinuous on U^*. Let γ be an index of the uniformity $\mathcal{U}$ of X such that $\mathrm{cl}(p\gamma)\subset U\cap U^*$ and $\mathrm{cl}(p\gamma)$ is compact. Let α be an index of X such that $p\alpha^2\subset p\gamma$, $\mathrm{cl}(p\alpha)$ is compact, $\mathrm{cl}(p\alpha)\alpha\subset U\cap U^*$.

Let $W=\{t|t\in T, pt\in p\alpha\}$. Then $p\overline{W}\subset \mathrm{cl}(pW)\subset \mathrm{cl}(p\alpha)\subset U$ and $\overline{W}$ is extensive but not syndetic. For $\overline{W}\subset A$ and if $\overline{W}$ were syndetic, A would be syndetic, which is not the case.

REMARK 4. If $K_{2n}=C\times[0, 2n]\times\cdots\times[0, 2n]$ then there exists a t in T such that $tK_{2n}\cap\overline{W}=\varnothing$.

PROOF. This is the negation of Remark 3 for the compact set K_{2n}.

REMARK 5. There exists a t' in T such that $\mathrm{int}(t'K_{2n})\cap\overline{W}=\varnothing$ and $\mathrm{Fr}(t'K_{2n})\cap\overline{W}\neq\varnothing$.

PROOF. This is accomplished by a translation of the set tK_{2n} obtained from Remark 4.

REMARK 6. There is a box $K_n'=C\times I_1^n\times\cdots\times I_{m+l}^n$ where $I_i^n=[0, n]$ or $[-n, 0]$, and an s_n in $\overline{W}$, such that $\mathrm{int}(s_nK_n')\cap\overline{W}=\varnothing$, $\mathrm{Fr}(s_nK_n')\cap\overline{W}\neq\varnothing$ and s_n appears at one corner of the box s_nK_n.

PROOF. This follows from Remark 5 by picking $s_n\in\mathrm{Fr}(t'K_{2n})\cap\overline{W}$ and taking a subbox of $t'K_{2n}$ whose sides are of length n.

The purpose of the last three remarks is to establish that there are boxes in $T=C\times I^l\times R^m$ of arbitrarily long sides with an element of $\overline{W}$ in one corner and no elements of $\overline{W}$ in the interior of the boxes. We will now establish the contradiction. The boxes above are constructed in such a way that we can distinguish one corner from the other. Hence we can classify each box by which corner the element of $\overline{W}$ appears. Since there are only 2^{m+l} corners there must be one corner in which the element of $\overline{W}$ appears for an infinite number of the boxes. We can assume without loss of generality that there are an infinite number of boxes s_kK_k' where $K_k'=C\times[0, k]\times\cdots\times[0, k]$ and $s_k\in\overline{W}$. Since $s_k\in\overline{W}$, ps_k is in $\mathrm{cl}(p\alpha)$ and therefore has a limit point q. Since $\mathrm{cl}(p\alpha)$ is compact there exists a subnet $\{N_m|m\in D\}$ of $\{ps_k\}$ such that $\{N_m|m\in D\}\to q$. Let β be an index of X such that $\beta\circ\beta\circ\beta\subset\alpha$ and, since q is an equicontinuous point, there is a neighborhood V of q such that x in V implies $(xt, qt)\in\beta$ for all t in T. Now there is an n in D such that ps_{k_m} is in V for all $m\geqq n$. From Remark 2 there is an r in T such that $p(s_{k_n}+r)$ is in $p\beta$ and every coordinate of r is positive. Hence

$$(p, p(s_{k_m}+r)) = (p, p(s_{k_n}+r))\circ(p(s_{k_n}+r), qr)\circ(qr, p(s_{k_m}+r))$$
$$\in\beta\circ\beta\circ\beta\subset\alpha,$$

so that $s_{k_m}+r\in\overline{W}$ for all $m\geqq n$. But if we choose an $m\geqq n$ such that k_m is strictly larger than the maximum coordinate of r then $s_{k_m}+r$ also belongs to $\mathrm{int}(s_{k_m}K_{k_m}')$, which is a contradiction. Therefore $\overline{W}$ is syndetic and p is an almost periodic point.

References

1. W. H. Gottschalk and G. A. Hedlund, *Topological dynamics*, Amer. Math. Soc. Colloq. Publ., vol. 36, Amer. Math. Soc., Providence, R.I., 1955. MR 17, 650.

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