A NOTE ON HOMOTOPIE EQUIVALENCES

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Abstract. Given a homotopy equivalence \( f: X \to Y \), a homotopy inverse \( g \) of \( f \), and a homotopy \( H: X \times I \to X \) from \( g \circ f \) to \( 1_X \). We show that there is a homotopy \( K: Y \times I \to Y \) from \( g \circ f \) to \( 1_Y \) such that \( f \circ H \simeq K \circ (f \times 1_I) \) rel \( X \times \partial I \) and \( H \circ (g \times 1_I) \simeq g \circ K \) rel \( Y \times \partial I \).

In [1], R. Lashof introduced the notion of a strong homotopy equivalence in context with the study of reductions of topological microbundles to piecewise linear or differentiable microbundles.

Definition. A strong homotopy equivalence between two spaces \( X \) and \( Y \) is a quadruple \((f, g, H, K)\) where \( f: X \to Y \) and \( g: Y \to X \) are maps and \( H: X \times I \to X \) and \( K: Y \times I \to Y \) are homotopies, \( H : g \circ f \simeq 1_X \), \( K : f \circ g \simeq 1_Y \), such that \( f \circ / \simeq / \) rel endpoints and \( H \circ (g \times 1_I) \simeq g \circ K \) rel endpoints (here \( I \) denotes the unit interval).

Now the question arises, whether any homotopy equivalence can be made into a strong one. The following proposition gives an affirmative answer. The result is also quite useful for many questions in homotopy theory. Although it is implicitly contained in various papers, I have never found it stated explicitly. The proof is very elementary, only using the basic facts about track addition of homotopies.

Proposition. Let \( f: X \to Y \) be a homotopy equivalence with homotopy inverse \( g \). Let \( H: X \times I \to X \) be a homotopy from \( g \circ f \) to \( 1_X \). Then there exists a homotopy \( K: Y \times I \to Y \) from \( g \circ f \) to \( 1_Y \) such that \( f \circ H \simeq K \circ (f \times 1_I) \) rel endpoints and \( H \circ (g \times 1_I) \simeq g \circ K \) rel endpoints.

Let \([X, Z]_p^q\) denote the set of equivalence classes of homotopies \( F: X \times I \to Z \) such that \( F|X \times 0 = p: X \to Z \) and \( F|X \times 1 = q: X \to Z \). Two such homotopies are defined to be equivalent if they are homotopic rel endpoints.

Maps \( h: X \to Y \) and \( k: Z \to W \) induce maps \( h^*: [Y, Z]_p^q \to [X, Z]_{p \circ h}^{q \circ h} \), given by \( h^*[M] = [M \circ (h \times 1)] \), where \([M]\) denotes the equivalence class of \( M \), and \( k_*: [X, Z]_u^w \to [X, W]_{k \circ u}^{k \circ w} \), given by \( k_*[N] = [k \circ N] \).

Lemma 1. Let \([R] \in [U, V]_0^1 \) and \([Q] \in [V, W]_u^w \). Then

\[
[u \circ R + Q \circ (l \times 1)] = [Q \circ (k \times 1) + v \circ R]
\]

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in \([U, W]^0\) where "+" denotes track addition, the left homotopy is applied first.

**Proof.** The map \(G: U \times I \times I \to W\), given by \(G(x, t_1, t_2) = Q(R(x, t_1), t_2)\) is the required homotopy rel endpoints between \(u \circ R + Q \circ (I \times 1)\) and \(Q \circ (k \times 1) + v \circ R\), because \(G|U \times 0 \times I = Q \circ (k \times 1), g|U \times I \times 1 = Q \circ (I \times 1)\), \(g|U \times I \times 0 = u \circ R, G|U \times I \times 1 = v \circ R\).

**Lemma 2.** With the notation of the proposition, \(f^* : [Y, Z]_u \to [X, Z]_u\) and \(g^* : [X, Z]_u \to [Y, Z]_u\) are bijections.

**Proof.** Consider the composite
\[
(*) \quad r: [Y, Z]^g \xrightarrow{f^*} [X, Z]_{\omega, f}^g \xrightarrow{g^*} [Y, Z]_{\omega, f}^s \xrightarrow{s} [Y, Z]^g
\]
where \(s\) is given by \(s[M] = [-p \circ F + M + q \circ F]\), and \(F: p \circ g \simeq 1_Y\) is an arbitrary fixed homotopy. For \([Q] \in [Y, Z]_u^g\), we have
\[
r[Q] = [-p \circ F + Q \circ (f \circ g \times 1) + q \circ F].
\]
Substituting \(R\) by \(F\) in the previous lemma, we find that \(r[Q] = [Q]\). Since \(s\) is a bijection (with inverse \([N] \mapsto [p \circ F + N - q \circ F]\)), the composite \(g^* \circ f^*\) is a bijection, and hence \(f^*\) injective and \(g^*\) surjective. Analogously one shows that in the sequence
\[
(**) \quad [X, Z]^g \xrightarrow{g^*} [Y, Z]_{\omega, f}^s \xrightarrow{s} [Y, Z]_{\omega, f}^g \xrightarrow{f^*} [X, Z]_{\omega, f}^g
\]
\(g^*\) is injective and \(f^*\) is surjective. Hence, in \((*)\), \(g^*\) and therefore \(f^*\) are bijective. Similarly in \((**)*\), \(f^*\) and hence \(g^*\) are bijective.

**Proof of the Proposition.** We are given \(H: g \circ f \simeq 1_X\). Choose \(K\) to be a representative of \(f^{-1}[f \circ H] \in [Y, Y]_{\omega, f}\), and let \(H'\) be a representative of \(g^{-1}[g \circ K] \in [X, X]_{\omega, f}\). Then \(K \circ (f \times 1) \simeq f \circ H\) rel endpoints, and \(H' \circ (g \times 1) \simeq g \circ K\) rel endpoints. Consider the composite
\[
r: [Y, X]_{\omega, f}^g \xrightarrow{f^*} [Y, Y]_{\omega, f}^s \xrightarrow{s} [Y, Y]_{\omega, f}^g \xrightarrow{g^*} [Y, X]_{\omega, f}^g
\]
where \(s\) is defined by \(s[M] = [-K \circ (f \circ g \times 1) + M + K]\). Let \([R] \in [Y, X]_{\omega, f}^g\), then
\[
r[R] = [-g \circ K \circ (f \circ g \times 1) + g \circ f \circ R + g \circ K] = [-H' \circ (g \circ f \circ g \times 1) + g \circ f \circ R + H' \circ (g \times 1)] = [R].
\]
The last equality follows from Lemma 1 by substituting \(Q\) by \(H'\).
Now \([H \circ (g \times 1)] \in [Y, X]_{\mathcal{P}(f \circ g)}\), and therefore

\[
[H \circ (g \times 1)] = r[H \circ (g \times 1)]
\]
\[
= [-g \circ K \circ (f \circ g \times 1) + g \circ f \circ H \circ (g \times 1) + g \circ K]
\]
\[
= [-g \circ f \circ g \times 1 + g \circ K \circ (f \circ g \times 1) + g \circ K]
\]
\[
= [g \circ K].
\]

So \(H \circ (g \times 1) \simeq g \circ K\) rel endpoints.

REFERENCES


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