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A NOTE ON HOMOTOPY EQUIVALENCES

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ABSTRACT. Given a homotopy equivalence $f:X\to Y$, a homotopy inverse g of f, and a homotopy $H:X\times I\to X$ from $g\circ f$ to 1_X. We show that there is a homotopy $K: Y\times I\to Y$ from $f\circ g$ to 1_Y such that $f\circ H\simeq K\circ(f\times 1_I)$ rel $X\times\partial I$ and $H\circ(g\times 1_I)\simeq g\circ K$ rel $Y\times\partial I$.

In [1], R. Lashof introduced the notion of a strong homotopy equivalence in context with the study of reductions of topological microbundles to piecewise linear or differentiable microbundles.

DEFINITION. A strong homotopy equivalence between two spaces X and Y is a quadruple (f, g, H, K) where $f:X\to Y$ and $g: Y\to X$ are maps and $H:X\times I\to X$ and $K: Y\times I\to Y$ are homotopies, $H:g\circ f\simeq 1_X$, K: $f\circ g\simeq 1_Y$, such that $f\circ H\simeq K\circ(f\times 1_I)$ rel endpoints and $H\circ(g\times 1_I)\simeq g\circ K$ rel endpoints (here I denotes the unit interval).

Now the question arises, whether any homotopy equivalence can be made into a strong one. The following proposition gives an affirmative answer. The result is also quite useful for many questions in homotopy theory. Although it is implicitly contained in various papers, I have never found it stated explicitly. The proof is very elementary, only using the basic facts about track addition of homotopies.

PROPOSITION. *Let $f:X\to Y$ be a homotopy equivalence with homotopy inverse g. Let $H:X\times I\to X$ be a homotopy from $g\circ f$ to 1_X. Then there exists a homotopy $K:Y\times I\to Y$ from $f\circ g$ to 1_Y such that $f\circ H\simeq K\circ(f\times 1_I)$ rel endpoints and $H\circ(g\times 1_I)\simeq g\circ K$ rel endpoints.*

Let $[X,Z]_p^q$ denote the set of equivalence classes of homotopies $F:X\times I\to Z$ such that $F|X\times 0=p:X\to Z$ and $F|X\times 1=q:X\to Z$. Two such homotopies are defined to be equivalent if they are homotopic rel endpoints.

Maps $h:X\to Y$ and $k:Z\to W$ induce maps $h^*:[Y,Z]_p^q\to[X,Z]_{p\circ h}^{q\circ h}$, given by $h^*[M]=[M\circ(h\times 1)]$, where $[M]$ denotes the equivalence class of M, and $k_*:[X,Z]_u^v\to[X,W]_{k\circ u}^{k\circ v}$, given by $k_*[N]=[k\circ N]$.

LEMMA 1. *Let $[R]\in[U,V]_k^l$ and $[Q]\in[V,W]_u^v$. Then*

$$[u\circ R+Q\circ(l\times 1)]=[Q\circ(k\times 1)+v\circ R]$$

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in $[U, W]^{v\circ l}_{u\circ k}$, *where* "+" *denotes track addition, the left homotopy is applied first.*

PROOF. The map $G: U\times I\times I\to W$, given by $G(x, t_1, t_2)=Q(R(x, t_1), t_2)$ is the required homotopy rel endpoints between $u\circ R+Q\circ(l\times 1)$ and $Q\circ(k\times 1)+v\circ R$, because $G|U\times 0\times I=Q\circ(k\times 1)$, $g|U\times 1\times I=Q\circ(l\times 1)$, $g|U\times I\times 0=u\circ R$, $G|U\times I\times 1=v\circ R$.

LEMMA 2. *With the notation of the proposition,* $f^*: [Y, Z]^q_p\to[X, Z]^{q\circ f}_{p\circ f}$ *and* $g^*: [X, Z]^v_u\to[Y, Z]^{v\circ g}_{u\circ g}$ *are bijections.*

PROOF. Consider the composite

$$(*) \qquad r: [Y, Z]^q_p \xrightarrow{f^*} [X, Z]^{q\circ f}_{p\circ f} \xrightarrow{g^*} [Y, Z]^{q\circ f\circ g}_{p\circ f\circ g} \xrightarrow{s} [Y, Z]^q_q$$

where s is given by $s[M]=[-p\circ F+M+q\circ F]$, and $F: f\circ g\simeq 1_Y$ is an arbitrary fixed homotopy. For $[Q]\in[Y, Z]^q_p$, we have

$$r[Q] = [-p\circ F + Q\circ(f\circ g\times 1) + q\circ F].$$

Substituting R by F in the previous lemma, we find that $r[Q]=[Q]$. Since s is a bijection (with inverse $[N]\mapsto[p\circ F+N-q\circ F]$), the composite $g^*\circ f^*$ is a bijection, and hence f^* injective and g^* surjective. Analogously one shows that in the sequence

$$(**) \qquad [X, Z]^v_u \xrightarrow{g^*} [Y, Z]^{v\circ g}_{u\circ g} \xrightarrow{f^*} [X, Z]^{v\circ g\circ f}_{u\circ g\circ f}$$

g^* is injective and f^* is surjective. Hence, in (*), g^* and therefore f^* are bijective. Similarly in (**), f^* and hence g^* are bijective.

PROOF OF THE PROPOSITION. We are given $H: g\circ f\simeq 1_X$. Choose K to be a representative of $f^{*-1}[f\circ H]\in[Y, Y]^1_{f\circ g}$, and let H' be a representative of $g^{*-1}[g\circ K]\in[X, X]^1_{g\circ f}$. Then $K\circ(f\times 1)\simeq f\circ H$ rel endpoints, and $H'\circ(g\times 1)\simeq g\circ K$ rel endpoints. Consider the composite

$$r: [Y, X]^g_{g\circ f\circ g} \xrightarrow{f_*} [Y, Y]^{f\circ g}_{f\circ g\circ f\circ g} \xrightarrow{s} [Y, Y]^1_{f\circ g} \xrightarrow{g_*} [Y, X]^g_{g\circ f\circ g}$$

where s is defined by $s[M]=[-K\circ(f\circ g\times 1)+M+K]$. Let $[R]\in[Y, X]^g_{g\circ f\circ g}$, then

$$\begin{aligned} r[R] &= [-g\circ K\circ(f\circ g\times 1) + g\circ f\circ R + g\circ K] \\ &= [-H'\circ(g\circ f\circ g\times 1) + g\circ f\circ R + H'\circ(g\times 1)] = [R]. \end{aligned}$$

The last equality follows from Lemma 1 by substituting Q by H'.

Now $[H\circ(g\times 1)]\in[Y, X]^{g}_{g\circ f\circ g}$, and therefore

$$\begin{aligned}[H\circ(g\times 1)] &= r[H\circ(g\times 1)]\\ &= [-g\circ K\circ(f\circ g\times 1)+g\circ f\circ H\circ(g\times 1)+g\circ K]\\ &= [-g\circ K\circ(f\circ g\times 1)+g\circ K\circ(f\circ g\times 1)+g\circ K]\\ &= [g\circ K].\end{aligned}$$

So $H\circ(g\times 1)\simeq g\circ K$ rel endpoints.

References

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