DARBOUX'S THEOREM FAILS FOR WEAK SYMPLECTIC FORMS

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Abstract. An example of a weak symplectic form on a Hilbert space for which Darboux’s theorem fails is given.

Introduction. Let $E$ be a Banach space and $B : E \times E \to \mathbb{R}$ a continuous bilinear form. Let $B^b : E \to E^*$ be defined by $B^b(e) \cdot f = B(e, f)$. Call $B$ nondegenerate if $B^b$ is an isomorphism and call $B$ weakly nondegenerate if $B^b$ is injective. For a symmetric bilinear form $G$ on $E$, define the skew form $\tilde{G}$ on $E \times E$ by

$$\tilde{G}((e_1, e_2), (f_1, f_2)) = G(f_2, e_1) - G(e_2, f_1).$$

It is easily seen that $\tilde{G}$ is nondegenerate (resp. weakly nondegenerate) iff $G$ is.

Now let $M$ be a Banach manifold. A symplectic form (resp. weak symplectic form) on $M$ is a smooth closed two form $\omega$ on $M$ such that for each $p \in M$, $\omega$ as a bilinear form on $T_p M$ is nondegenerate (resp. weakly nondegenerate); here $T_p M$ is the tangent space at $p$. Using a technique of Moser, Weinstein ([6], [7]) showed that for each $p \in M$ there is a local chart about $p$ on which $\omega$ is constant. This is a significant generalization and simplification of the classical theorem of Darboux. However, in many physical examples (the wave equation and fluid mechanics for instance) one deals with weak symplectic forms (see [1], [3], [4], [5]).

It is therefore interesting to know if Darboux’s theorem remains valid for weak symplectic forms. In this note we give a counterexample.

Symplectic forms induced by metrics. If $M$ is a manifold, its cotangent bundle $T^*M$ carries a canonical symplectic form $\omega$. If $M$ is modeled on a reflexive space the form is nondegenerate; otherwise it is only weakly nondegenerate. See [1], [4]. Now let $\langle \cdot, \cdot \rangle_p$ be a (smooth) weak riemannnian metric on $M$. Then it induces a map of $TM$ to $T^*M$. The pull back $\Omega$ of $\omega$ to $TM$ is called the form induced by the metric. It is a weak symplectic form.
form and in a chart $U$ for $M$ it is given by (using principal parts):

$$2\Omega_{u,e}((e_1, e_2), (e_3, e_4)) = Du(e, e_1) \cdot e_3 - Du(e, e_3) \cdot e_1 + \langle e_4, e_5 \rangle_{u} - \langle e_2, e_3 \rangle_{u}.$$  

Here, $Du$ denotes the derivative of the map $u \mapsto \langle e, e_1 \rangle_u$ with respect to $u$. In the finite dimensional case this corresponds to the classical formula

$$\Omega = \sum g_{ij} \, dq^i \wedge dq^j + \sum \frac{\partial g_{ij}}{\partial q^k} \, q^i \, dq^i \wedge dq^k.$$ 

Observe that in the finite dimensional case if we take new variables $q^1, \ldots, q^n, p_1, \ldots, p_n$ where $p_i = \sum g_{ij} q^j$, then (as is easy to check)

$$\Omega = \sum dq^i \wedge dp_i$$

which gives a chart in which $\Omega$ is constant.

**The example.** The following is a simplification of an earlier example. We thank the referee and Paul Chernoff for suggestions in this regard.

Let $H$ be a real Hilbert space. Let $S: H \to H$ be a compact operator with range a dense, but proper subset of $H$, which is selfadjoint and positive: $\langle Sx, x \rangle > 0$ for $0 \neq x \in H$. For example if $H = L^2(R)$, we can let $S = (1 - \Delta)^{-1}$ where $\Delta$ is the Laplacian; the range of $S$ is $H^2(R)$.

Since $S$ is positive, $-1$ is clearly not an eigenvalue. Thus, by the Fredholm alternative, $aI + S$ is onto for any real scalar $a > 0$. Define on $H$ the weak metric $g(x)(e, f) = \langle A_x e, f \rangle$ where $A_x = S + \|x\|^2 I$. Clearly $g$ is smooth in $x$, and is an inner product. Let $\Omega$ be the weak symplectic form on $H \times H = H_1$ induced by $g$, as was discussed above.

**PROPOSITION.** There is no coordinate chart about $(0, 0) \in H_1$ on which $\Omega$ is constant.

**PROOF.** If there were such a chart, say $\phi: U \to H \times H$ where $U$ is a neighborhood of $(0, 0)$, then in particular in this chart, the range $F$ of $\Omega^b$, as a map of $H_1$ to $H^*_1$, would be constant. Let $B_{x,y}$ be the derivative of $\phi$ at $(x, y) \in H_1$. Then we obtain that the range of $\Omega^b_{x,y}$ equals $B_{x,y}^* F$.

Now by the above formula for $\Omega$, at the point $(x, 0)$ we have

$$2\Omega_{(x,0)}((e_1, e_2), (e_3, e_4)) = g_x(e_4, e_1) - g_x(e_2, e_3).$$

But by construction, for $x \neq 0$, $g_x$ is a strong metric (i.e., $A_x$ is onto for $x \neq 0$), so the range of $\Omega^b_{(x,0)}$ is all of $H^*_1$ for $x \neq 0$. Since $B_{x,y}$ is an isomorphism, this implies that $\Omega^b_{(0,0)}$ is onto all of $H^*_1$ as well. But $g_0$ is only a weak metric which is not onto as a map of $H_1$ to $H^*_1$. Hence $\Omega^b_{(0,0)}$ cannot be onto as well, a contradiction.

As was pointed out by the referee, the example even shows that $\Omega$ cannot be made constant on a continuous vector bundle chart on $T^2 M \to TM$, let alone by a manifold chart on $TM$. 

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Of course the essence of the example is that the range of $\Omega$ suddenly changed at one point i.e., the topology of the metric suddenly changed. This is perfectly compatible with the smoothness of $\Omega$ as it is only a weak symplectic form. This suggests a possible conjecture pointed out by Paul Chernoff: If $\Omega$ is such that the ranges of $\Omega_u$ are locally equivalent via an isomorphism, then Darboux's theorem should hold. This can be verified directly in case $\Omega$ comes from a metric which has locally equivalent ranges.

REFERENCES

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