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A CHARACTERIZATION OF GENERAL Z.P.I.-RINGS[1]

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ABSTRACT. A commutative ring A is a general Z.P.I.-ring if each ideal of A can be represented as a finite product of prime ideals. We prove that a commutative ring A is a general Z.P.I.-ring if each finitely generated ideal of A can be represented as a finite product of prime ideals. We also give a characterization of Krull domains in terms of *-operations, as defined by Gilmer.

Introduction. A commutative ring A is a *π-ring* if each principal ideal of A can be represented as a product of prime ideals; A is a *p-ring* if each finitely generated ideal of A can be represented as a product of prime ideals. A commutative ring A is a *general Z.P.I.-ring* if each ideal of A can be represented as a product of prime ideals. Mori characterized π-rings in **[4]**, **[5]** and general Z.P.I.-rings in **[6]**. In **[9]** Wood extended Mori's results on π-rings. Wood also gave a characterization of general Z.P.I.-rings which is independent of Mori's work **[10]**.

Let D be an integral domain with identity and with quotient field K. If $F(D)$ is the set of nonzero fractional ideals of A, a mapping $B \to B^*$ of $F(D)$ into $F(D)$ is called a **-operation on D* if the following three conditions hold for any a in $K-\{0\}$, and any B, C in $F(D)$.

(1) $(a)^*=(a)$, $(aB)^*=aB^*$.

(2) $B \subseteq B^*$, if $B \subseteq C$, $B^* \subseteq C^*$.

(3) $(B^*)^*=B^*$.

If $(B)^*=(C)^*$, B and C are called **-equivalent*, denoted $B \sim_* C$. If $B=B^*$, B is called a **-ideal*. An integral domain D is a Krull domain if $D=\bigcap V_\alpha$, where $\{V_\alpha\}$ is a set of rank one discrete valuation rings with the property that for each nonzero x in D, $xV_\alpha=V_\alpha$ for all but a finite number of the V_α.

In the first section of this paper, we give a new characterization of Krull domains. We prove that an integral domain A with identity is a Krull domain if and only if there is a *-operation on A such that each nonzero principal ideal of A is *-equivalent to a product of prime ideals. Then we consider π-rings with no zero divisors, called *π-domains*. Theorem 1.2 states that A is a π-domain with identity if and only if A is a Krull domain

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in which each minimal prime ideal is invertible. We also prove that if A is a π-domain with identity, then each generalized quotient ring of A is also a π-domain with identity.

In §2, we consider p-rings, as well as *p-domains*, those p-rings with no zero divisors. Clearly, any Dedekind domain is a p-domain with identity. In Theorem 2.1, we show that any p-domain with identity is a Dedekind domain. Using Theorem 2.1, we prove the main result of this section, Theorem 2.3. This theorem states that a necessary and sufficient condition for a ring A to be a general Z.P.I.-ring is that A be a p-ring.

1. **π-rings.** Let A be an integral domain with identity. An example of a *-operation on A is the *v-operation* which associates with each nonzero fractional ideal B of A the intersection of all the principal fractional ideals of A containing B. This intersection is denoted *div*(B); div(B) is called a *divisorial ideal.*

THEOREM 1.1. *Let A be an integral domain with identity. A is a Krull domain if and only if there exists a *-operation on A such that each nonzero principal ideal of A is *-equivalent to a product of prime ideals.*

PROOF. ($\rightarrow$) Let A be a Krull domain. By [**7**, Theorem 1], $v: B \rightarrow \text{div}(B)$ is a *-operation on A such that each nonzero principal ideal of A is v-equivalent to a product of prime ideals of A.

($\leftarrow$) Let $a \in A - \{0\}$. $(a) \sim_* C$, where C is a finite product of prime ideals of A. By [**3**, Theorem 28.1], $C \subseteq (a) \subseteq \text{div}(C)$. It follows that $(a) \sim_v \text{div}(C)$. Since (a) is a divisorial ideal, $(a) = \text{div}(C)$. By [**8**, Theorem 3.1], A is a Krull domain.

If D is an integral domain, we call a prime ideal P of D a *minimal prime* of D if P is of height 1.

THEOREM 1.2. *Let A be an integral domain with identity. A is a π-domain if and only if A is a Krull domain in which each minimal prime ideal is invertible.*

PROOF. ($\rightarrow$) Since each nonzero principal ideal of A can be represented as a product of prime ideals, each nonzero principal ideal of A is v-equivalent to a product of prime ideals. This implies that A is a Krull domain by Theorem 1.1. That each minimal prime ideal of A is invertible follows from [**10**, Theorem 1.1.3].

($\leftarrow$) Let d be a nonzero nonunit of A. (d) can be represented as a finite intersection of primary ideals belonging to minimal primes of A because A is a Krull domain [**3**, Corollary 35.10]. Let $(d) = \bigcap_{i=1}^{n} Q_i$, where for each i, Q_i is P_i-primary and P_i is a minimal prime ideal of A. Since each P_i is invertible, $Q_i = P_i^{e_i}$ for some nonnegative integer e_i [**2**, Theorem 6.6].

Hence $(d)=\bigcap_{i=1}^n P_i^{e_i}$, where for each i, P_i is a minimal prime of A. We can assume that all the P_i are distinct. Then by [**3**, Lemma 35.15],

$$(d)=\bigcap_{i=1}^{n} P_i^{e_i}=\prod_{i=1}^{n} P_i^{e_i}.$$

Hence A is a π-domain.

Let A be an integral domain with quotient field K. If A' is a ring such that $A\subset A'\subset K$, A' is an overring of A. Let A be an integral domain with identity and with quotient field K. Let B be an overring of A. B is called a *generalized quotient ring of A* if B is flat as an A-module.

THEOREM 1.3. *Let A be a π-domain with identity. Each generalized quotient ring of A is a π-domain.*

PROOF. Let B be a generalized quotient ring of A. Each prime ideal of A extends in B either to B or to a proper prime ideal of B [**1**, Theorem 2]. If $b\in B-\{0\}$, $b=a_1/a_2$, where a_1, a_2 are in $A-\{0\}$. Since a_1A and a_2A can be represented as finite products of prime ideals of A, bA can also be represented as a finite product of prime ideals of A; $bA=\prod_{i=1}^n P_i^{e_i}$ where for each i, P_i is a prime ideal of A and $e_i\in Z-\{0\}$. It follows that

$$bB=\left(\prod_{i=1}^{n} P_i^{e_i}\right)B=\prod_{i=1}^{n}(P_iB)^{e_i},$$

which is a representation of bB as a product of prime ideals of B.

2. *p*-**rings.** We recall that a *Dedekind domain* is an integral domain with identity such that each ideal is representable as a product of prime ideals. A principal ideal ring A with identity is called a *special primary ring* if A contains only one prime ideal $M\neq A$ and if $M^k=(0)$ for some positive integer k.

THEOREM 2.1. *Let A be an integral domain with identity and with quotient field K. A is a Dedekind domain if and only if A is a p-domain.*

PROOF. $(\rightarrow)$ By definition of Dedekind domain.

$(\leftarrow)$ Every p-ring is a π-ring. By Theorem 1.2, A is a Krull domain in which each minimal prime ideal is invertible. To conclude that A is Dedekind it suffices for us to prove that A is of Krull dimension 1. Each prime ideal of a Krull domain contains a minimal prime ideal. It follows that if each minimal prime ideal is maximal, A is Dedekind.

Let P be any minimal prime ideal of A. Let $a\in A-P$. Since (P,a) is finitely generated, $(P,a)=\prod_{i=1}^n P_i$ and $(P,a^2)=\prod_{j=1}^m Q_j$, where for each i and j, P_i and Q_j are prime ideals of A. Let $\bar{A}=A/P$, and let $\bar{a}$ be the residue class of a modulo P. Then $(\bar{a})=\prod_{i=1}^n (P_i/P)$, and $(\bar{a}^2)=\prod_{j=1}^m (Q_j/P)$. Since $(\bar{a})$ is an invertible ideal of $\bar{A}$, $\{P_i/P\}_{i=1}^n$ and $\{Q_j/P\}_{j=1}^m$ are invertible prime ideals of $\bar{A}$. By [**11**, Lemma 5, p. 272], $m=2n$, and the Q_j's can be renumbered in such a way that

$$Q_{2i}/P = Q_{2i-1}/P = P_i/P.$$

Thus $Q_{2i}=Q_{2i-1}=P_i$, and $(P, a^2)=(P, a)^2$. Now $P\subset(P, a)^2\subset(P^2, a)$. For any $x\in P$, $x=y+za$ where $y\in P^2$ and $z\in A$. Since $za\in P$, $z\in P$. This implies that $P\subset P^2+Pa$. The reverse inclusion, $P^2+Pa\subset P$, is clear. It follows that

$$P = P^2 + Pa = P(P + (a)).$$

Recalling that P is invertible, we conclude that $A=P+(a)$, and that P is a maximal ideal. Therefore, each minimal prime ideal is a maximal ideal.

LEMMA 2.2. *Let A be a p-ring. If B is any finitely generated ideal of A, then A/B is a p-ring.*

PROOF. Let $\bar{C}$ be a finitely generated proper ideal of A/B. $\bar{C}$ is the image of a finitely generated ideal C which contains B. $C=\prod_{i=1}^{n} P_i^{e_i}$, where for each i, P_i is a prime ideal of A containing B. Then $\bar{C}=\prod_{i=1}^{n}(P_i/B)^{e_i}$.

Using Theorem 2.1 and Lemma 2.2 we now prove the main result of this paper.

THEOREM 2.3. *A is a general Z.P.I.-ring if and only if A is a p-ring.*

PROOF. ($\rightarrow$) By definition of general Z.P.I.-ring.

($\leftarrow$) We consider three cases: (a) A is a commutative ring with identity, (b) A is a commutative ring without identity, but with zero divisors; and (c) A is an integral domain without identity.

(a) A is a finite direct sum of π-domains with identity and special primary rings by [**4**, Hauptsatz]. Using [**9**, Theorem 2], we can conclude A is a general Z.P.I.-ring if any summand A_j of A that is a domain is Dedekind. Let $A=\sum_{i=1}^{n} A_i$ be a representation of A as a direct sum of π-domains with identity and special primary rings. If e_i is the identity of the summand A_i, $A_i=Ae_i$. Thus for any j, A_j is isomorphic to $A/(\{e_i\}_{i\neq j})$. By Lemma 2.2, A_j is a p-ring. By Theorem 2.1, if A_j is a domain, A_j is Dedekind.

(b) By [**5**, Hauptsatz II], $A=M$ or $A=M+K$, where K is a field and M is a ring without identity such that each ideal of M is a power of M. A is a general Z.P.I.-ring by [**9**, Theorem 2].

(c) It is clear from the definition of a p-ring that a Noetherian p-ring is a general Z.P.I.-ring. We now show that if the p-ring A is an integral domain without identity, then A is Noetherian. To prove that A is Noetherian we show that for any nonzero element $d\in A$, $A/(d^2)$ is Noetherian. If $d\in A-\{0\}$, $A/(d^2)$ is a commutative ring with zero divisors. By Lemma 2.2, $A/(d^2)$ is a p-ring. Since $A/(d^2)$ is contained in either the class of p-rings considered in (a) or the class considered in (b), $A/(d^2)$ is a general Z.P.I.-ring.

All general Z.P.I.-rings are Noetherian by [**10**, Theorem 2.1.21]. Hence $A/(d^2)$ is Noetherian.

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