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A NORMAL SUBGROUP OF A SEMISIMPLE LIE GROUP IS CLOSED

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ABSTRACT. The theorem of the title is proved.

It is common to relate the definition of "simple" as applied to discrete groups to the definition of "simple" as applied to Lie groups in the following way. A discrete group is simple if every proper normal subgroup is trivial, while a connected Lie group is simple if every proper *closed* normal subgroup is trivial as a Lie group, i.e. is discrete. We seem to have heard that the two uses can be brought closer together by omitting the word "closed" but we have not found a proof of this in print. In this short note we shall show that every normal subgroup of a connected semisimple Lie group is closed, so that the desired omission can be made.

We need one piece of notation. For any group G and a in G we let $C^a=\{gag^{-1}:g \text{ in } G\}$ be the *conjugacy class* of a. Then we have the following key

LEMMA. *Let G be a connected simple Lie group. If a in G is noncentral, then the product set $(C^a)^{\dim G}$ has nonempty interior.*

PROOF. Let $n=\dim G$. Then the lemma follows immediately from the fact that the map $f(g_1, \cdots, g_n)=g_1ag_1^{-1}\cdots g_nag_n^{-1}$, of $G\times G\times\cdots\times G$ (n factors) to G, has rank n at some point. This is a consequence of Theorem 1.1 of [2], where the simple details involved in computing the differential of f are worked out. $\square$

From this lemma we can show that the two definitions of "simple" can be more closely related.

COROLLARY. *Let N be any algebraically normal subgroup of a connected simple Lie group G. Then either $N=G$ or $N\subseteq Z=$center of G. Hence N is closed.*

PROOF. Since $(C^a)^{\dim G}\subseteq N$, for any a in N, we see that either $N\subseteq Z$ or N is open. But N open implies $N=G$ by connectedness. Since Z is discrete in G, N is always closed. $\square$

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For semisimple groups we can extend our arguments to get a rather detailed description of all normal subgroups.

THEOREM. *Let N be an algebraically normal subgroup of a connected semisimple Lie group G. Then N is closed. More specifically, let $\mathfrak{g}=\mathfrak{g}_1+\cdots+\mathfrak{g}_k$ be the decomposition of the Lie algebra of G into simple ideals, and for each i, $i=1,\cdots,k$, let G_i be the connected normal simple Lie subgroup of G with Lie algebra $\mathfrak{g}_i$. Then $N=G_{i_1}\cdot\cdots\cdot G_{i_l}\cdot D$, for some subgroup, D, of the center of G and some i_j, $1\leqq i_1<\cdots<i_l\leqq k$.*

PROOF. The case $k=1$ is just the previous corollary. We proceed by induction. Note that $G=G_1\cdot\cdots\cdot G_n$ since the right side is a connected subgroup of G with Lie algebra $\mathfrak{g}$. Now if N is included in the center of G we are done. So suppose some n in N has a representation in the form $n=a_1\cdot\cdots\cdot a_n$ with a_i in G_i, and some a_i, say a_1, noncentral. We shall show that $G_1\cap N=G_1$. Since G_i and G_j commute for $i\neq j$, we see that a_1 must be noncentral in G_1 and C^{a_1} is equal to the conjugacy class of a_1 in G_1. If $S=\{g_1ng_1^{-1}:g_1 \text{ in } G_1\}=C^{a_1}\cdot a_2\cdots a_n$, then the previous lemma shows that $S^{\dim G_1}n^{-\dim G_1}=(C^{a_1})^{\dim G_1}a_1^{-\dim G_1}\subseteq G_1$ has nonempty interior relative to G_1. Since N is normal, $S^{\dim G_1}\cdot n^{-\dim G_1}\subseteq N$, so the subgroup $N\cap G_1$ has nonempty interior relative to G_1. Since G_1 is connected, we have $G_1=G_1\cap N$.

Now, from the theorem of §6 in [**1**], we see that G_1 is closed in G, so the natural projection $\sigma:G\to G/G_1$ is continuous. Since $\sigma(N)$ is normal in G/G_1, the induction hypothesis implies $\sigma(N)$ is closed. Thus $N=\sigma^{-1}(\sigma(N))$ is closed. Moreover, $\sigma(N)=\sigma(G_{i_2})\cdot\cdots\cdot\sigma(G_{i_l})\cdot D'$ for some subgroup, D', of the center of G/G_1 and some i_j, $2\leqq i_2<\cdots<i_l\leqq n$. Since σ maps the center of G *onto* the center of G/G_1, $\sigma^{-1}(D')=G_1\cdot D$ for some subgroup, D, of the center of G. Thus $N=G_1\cdot G_{i_2}\cdot\cdots\cdot G_{i_l}\cdot D$ as desired. □

We remark that the appeal to Mostow's paper [**1**] can be avoided for semisimple groups of the form $G=G_1\times\cdots\times G_n/D$ where G_i is simple and D is a *finite* subgroup of the center of $G_1\times\cdots\times G_n$, since for such groups it is easy to see that the image of G_i is closed in G. All semisimple groups with finite center have this form, as do all simply connected groups.

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