PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 32, Number 2, April 1972

ON THE BEHAVIOR OF SOLUTIONS OF SUBLINEAR SECOND ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT. The equations in question generalize

$$x'' + a(t)\,|x|^{\gamma}\operatorname{sgn} x = 0, \qquad 0 < \gamma < 1, \qquad a(t) \geqq 0.$$

A comparison theorem and a uniqueness theorem for initial value problems are proved. Boundary value problems are studied. Oscillation is discussed via comparison.

1. **Introduction.** This paper deals with equations

$$x'' + F(t, x) = 0, \qquad xF \geqq 0, \tag{1}$$

which generalize

$$x'' + a(t)\,|x|^{\gamma}\operatorname{sgn} x = 0, \qquad 0 < \gamma < 1, \tag{2}$$

where $a(t)$ is nonnegative and continuous on $[0, \infty)$.

The principal results of the paper include an analog of a result of Moore and Nehari [10] about the solutions of the equation (2), $\gamma>1$, with boundary conditions $x(a)=x(b)=0$, $0<a<b<\infty$, having any given number of zeros in (a, b); an existence and uniqueness theorem for positive solutions of a boundary value problem; and an oscillation theorem.

Most of the arguments are based on a comparison theorem proved here. It is based on a theorem of Grimmer and Waltman [4] and is analogous to the classical Sturm Comparison Theorem. A uniqueness theorem for initial value problems related to a result of Belohorec [3] is also proved. Other studies of (2) include Heidel's [6], [7], [8], and Heidel and Hinton's [9].

2. **Comparison.** Theorem 1 is for a somewhat different nonlinearity than was considered by Grimmer and Waltman in [4] and it partially improves that result in that it holds for solutions of (2) which initiate on the t-axis. Note that Theorem 1 also holds for (2) with $a(t)\leqq 0$ and $\gamma>1$.

Received by the editors June 4, 1971.

AMS 1970 *subject classifications.* Primary 34C15, 34B15; Secondary 34A10.

Key words and phrases. Comparison, uniqueness, second order, nonlinear, initial value problem, boundary value problem, zeros, oscillation, singular solutions, positive solutions, Emden-Fowler equation.

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THEOREM 1. *Consider the inequalities*

$$x'' + F(t, x) \leqq 0, \tag{3}$$

$$y'' + F(t, y) \geqq 0, \tag{4}$$

where $F(t, u)$ is continuous for t, $u \geqq 0$ and $y \geqq x \geqq 0$ implies

$$yF(t, x) - xF(t, y) \geqq 0. \tag{5}$$

Let $x(t)$ satisfy (3) *and $y(t)$ satisfy* (4) *on $[a, b)$ with $x(t) > 0$ on (a, b). Suppose $y(a) = x(a) \geqq 0$ and $y'(a) \geqq x'(a)$ but not $x(a) = x'(a) = 0$.*

(i) *If any one of $y'(a) > x'(a)$ or* (3) *or* (4) *is strict at $t = a$ then $y(t) > x(t)$ on (a, b).*

(ii) *If $y'(a) = x'(a)$ and* (3) *and* (4) *are equations at $t = a$, let there be a null sequence ε_n of positive terms such that*

$$z'' + F(t, z) \mp \varepsilon_n = 0, \qquad n = 1, 2, \cdots, \tag{1_n}$$

has for each n and the given initial conditions a solution which exists on $[a, b)$. Then, uniqueness of the solution of (1) *for the given initial conditions implies $y(t) \geqq x(t)$ on $[a, b)$.*

PROOF. Multiply (3) by $y(t)$, (4) by $x(t)$, subtract, and integrate to obtain

$$\begin{aligned} &(y(t)x'(t) - x(t)y'(t)) + (-y(a)x'(a) + x(a)y'(a)) \\ &\qquad + \int_a^t (y(s)F(s, x(s)) - x(s)F(s, y(s)))\, ds \leqq 0. \end{aligned} \tag{6}$$

If $y'(a) > x'(a)$ and $t = c$ is the first point of intersection of $x(t)$, $y(t)$, in (a, b) then $x'(c) \geqq y'(c)$ and (6) is contradicted.

If (3) or (4) is strict then $y''(a) > x''(a)$ and a continuity argument leads as in the foregoing to a contradiction of the strictness of (6). Case (i) is established.

In case (ii) compare (essentially as in [**1**, pp. 80–81]) (3) and (4) respectively to

$$z_n'' + F(t, z_n) \mp \varepsilon_n = 0, \qquad n = 1, 2, \ldots,$$

where $z_n(a) = x(a)$, $z_n'(a) = x'(a)$. Then, as in case (i), $z_n^-(t) \geqq x(t)$ and $z_n^+(t) \leqq y(t)$. The uniqueness assumption ensures $z_n^-(t) - z_n^+(t) \to 0$ as $n \to \infty$ and the theorem is proved.

There is an obvious counterpart to Theorem 1 if $0 \geqq x \geqq y$ and the inequalities (3) and (4) are reversed and (5) holds. In the remainder of the paper (5) is assumed valid for both $y \geqq x \geqq 0$ and $0 \geqq x \geqq y$.

3. **Uniqueness.** Theorem 2 is suggested by a uniqueness result for (2) in [**3**].

THEOREM 2. *Given* (1) *let* $F(t, u)$ *be continuous on* $[0, \infty)\times(-\infty, \infty)$. *Suppose* $y>x>0$ *and* $0>x>y$ *imply* (5) *is strict except, possibly, for t-values in a set of measure zero. And, suppose, for each* t, $F(t, u)$ *is a nondecreasing function of* u. *Then, solutions of the initial value problem for* (1) *are unique if the trivial solution of* (1) *is unique.*

PROOF. Suppose $x(T)=y(T)\geqq 0$, $x'(T)=y'(T)$, and if $x(T)=0$ it may be supposed that $x'(T)>0$.

If $x(t)$ and $y(t)$ oscillate relatively in each interval $[T, T_1)$ then there is a pair a, b, such that $y(a)=x(a)$, $y(b)=x(b)$, and $y(t)>x(t)>0$ on (a, b). An inequality like (6) can be produced with the first quantity evaluated at b nonnegative, the second quantity evaluated at a nonnegative, and the integral over $[a, b]$ positive. This is a contradiction. Thus, suppose $y(t)>x(t)$ on some interval (T, T_1).

In the latter case (1) leads to

$$y(t) - x(t) = \int_T^t \int_T^s (F(r, x(r)) - F(r, y(r)))\, dr\, ds \leqq 0$$

which is, for $t>T$, an obvious contradiction. The theorem is proved.

REMARK 1. If $uF(t, u)\geqq 0$ and $x(T_0)=x'(T_0)=0$ and $x(t)$ is not the trivial solution then $x(t)$ must oscillate infinitely often in each neighborhood of T_0. In this case $x(t)$ is called a singular solution of (1). Heidel [7] gave an example of (2) with $a(t)$ positive and continuous having a singular solution. It is shown in [11] that (2) has no singular solutions if $a(t)$ is positive and of finite variation on finite intervals.

4. **Boundary value problems.** In this and the following section solutions of (1) are assumed to exist on sufficiently large intervals. Note that in the case of equation (2), given any $T\geqq 0$, a theorem of Wintner [5, p. 29] ensures every solution $x(t, T, x_0, x_1)$ exists on $[T, \infty)$.

Throughout this section F is assumed to be continuous on $[0, \infty)\times(-\infty, \infty)$, to satisfy condition (5), and, also,

$$xF(t, x) \geqq a_1(t)\,|x|^{\rho+1} \geqq 0, \tag{7}$$

$$a_2(t)\,|x|^{\sigma+1} \geqq xF(t, x), \qquad |x| \geqq X_0 > 0, \tag{8}$$

where $0<\rho\leqq\sigma<1$ and continuous $a_1(t)$, $a_2(t)$ only vanish on a set of measure zero.

The next lemma requires this general result: If $f(t)>0$, $f'(t)>0$, $f''(t)<0$, on (t_0, ∞) then

$$f(t) \geqq \tfrac{1}{2}tf'(t), \qquad t \geqq 2t_0. \tag{9}$$

Since $f'(t)$ decreases on (t_0, ∞) the result follows from an integration of $f'(t)$ on $[t_0, t]$.

LEMMA 1. *Given any* $T, x_0 \geqq 0$ *there exists an* $\eta(T, x_0) > 0$ *such that* $0 < x_1 < \eta$ *implies* $x(t, T, x_0, x_1)$ *has a zero in* (T, ∞).

PROOF. Suppose $x_0 > 0$. Then $x(t) \geqq x_0$ and (7) gives $x''(t) \leqq -a_1(t)x_0^\rho$. An integration makes clear for sufficiently small x_1 that $x'(t)$ has a zero and the result follows from the convexity of $x(t)$ and Theorem 1.

If $x_0 = 0$ suppose every solution is positive on (T, ∞). Then, $x(t)$ satisfies the inequality (9) for $t \geqq T_1 = 2T$. Hence, on $[T_1, \infty)$, (7) and (9) give

$$x''(t) + 2^{-\rho}a_1(t)t^\rho(x'(t))^\rho \leqq 0.$$

This inequality implies

$$(1-\rho)^{-1}(x'(t))^{1-\rho} \leqq (1-\rho)^{-1}(x'(T_1))^{1-\rho} - 2^{-\rho}\int_{T_1}^t s^\rho a_1(s)\,ds.$$

If x_1 is small enough $x'(t)$ must have a zero and thus the conclusion is as before. The lemma is proved.

If $x_0 \geqq 0$ and $x_1 > 0$ denote by $T(x_1)$ the first zero of a solution $x(t)$ of (1) to the right of T and by $X(x_1)$ the maximum of $x(t)$ on $(T, T(x_1))$. Lemma 1 ensures that if x_1 is small enough then $T(x_1)$ and $X(x_1)$ exist. Theorem 1 makes the next lemma obvious.

LEMMA 2. *If* $x_0 = 0$; *then*, $x_1 \to 0$ *implies* $X(x_1) \to 0$.

LEMMA 3. *If* $x_0 = 0$; *then*, $x_1 \to 0$ *implies* $T(x_1) \to T$.

PROOF. Let $T'(x_1)$ be the point of maximum. Integrate (1) twice to obtain, by (7),

$$X(x_1) \geqq \int_T^{T'(x_1)} \int_s^{T'(x_1)} a_1(r)(x(r))^\rho\,dr\,ds.$$

Since $x''(t) \leqq 0$, $x(t) \geqq X(x_1)(T'(x_1) - T)^{-1}(t - T)$, $T \leqq t \leqq T'(x_1)$. And, Theorem 1 implies for all small x_1 that $T'(x_1) - T \leqq \tau$ for some $\tau > 0$. Thus,

$$(X(x_1))^{1-\rho} \geqq \tau^{-\rho}\int_T^{T'(x_1)} \int_s^{T'(x_1)} a_1(r)(r - T)^\rho\,dr\,ds.$$

A similar argument gives

$$(X(x_1))^{1-\rho} \geqq \tau^{-\rho}\int_{T'(x_1)}^{T(x_1)} \int_{T'(x_1)}^{s} a_1(r)(T(x_1) - r)^\rho\,dr\,ds.$$

The result which is sought follows from these inequalities and Lemma 2 is an obvious way.

LEMMA 4. *If* $x_0 = 0$; *then*, $x_1 \to 0$ *implies* $x'(T(x_1)) \to 0$.

PROOF. Integrate (1) to get

$$-x'(T(x_1)) = \int_{T'(x_1)}^{T(x_1)} F(s, x(s))\, ds;$$

and the conclusion is a clear consequence of Lemmas 2 and 3.

LEMMA 5. *If all solutions of the initial value problem for* (1) *are unique, and* $x_0 \geqq 0$; *then, for some* $x_1^0 > 0$, $x_1 \to x_1^0 -$ *implies* $T(x_1) \to \infty$.

PROOF. Let $x_1^0 = \sup x_1$ such that $T(x_1)$ exists. Suppose $x_1^0 < \infty$ and the $T(x_1)$ are bounded. Then continuous dependence of solutions on x_1 implies the solution with initial slope x_1^0 both has a zero and is positive on (T, ∞), a contradiction.

Therefore, suppose $x_1^0 = \infty$. An integration of (1) gives

$$x_1 = \int_T^{T'(x_1)} F(s, x(s))\, ds;$$

and it is evident $x_1 \to \infty$ implies $T'(x_1) \to \infty$ and/or $X(x_1) \to \infty$.

In the latter case if $T'(x_1)$ has a positive upper bound let $T_0(x_1)$ be the first t-value after T at which $x(t) = X_0$. Then, integration of (1) gives, by use of (7),

$$X(x_1) \leqq \int_T^{T_0(x_1)} \int_s^{T'(x_1)} F(r, x(r))\, dr\, ds + \int_{T_0(x_1)}^{T'(x_1)} \int_s^{T'(x_1)} a_2(r) x^\sigma(r)\, dr\, ds.$$

Hence, $x(s) \leqq X(x_1)$ implies

$$(X(x_1))^{1-\sigma} \leqq (X(x_1))^{-\sigma} \int_T^{T_0(x_1)} \int_s^{T'(x_1)} F(r, x(r))\, dr\, ds$$
$$+ \int_{T_0(x_1)}^{T'(x_1)} \int_s^{T'(x_1)} a_2(r)\, dr\, ds.$$

Since it is assumed that both $T'(x_1)$ is bounded and $X(x_1) \to \infty$, the right side of this inequality is bounded. A contradiction results and the lemma is proved.

In **[10]** Moore and Nehari showed by a variational argument for the case $a(t)$ positive, $\gamma = 2n+1$, $n = 1, 2, \cdots$, in (2) that for any a, b, $0 < a < b < \infty$, and for each integer $m \geqq 0$, there exists a solution of (2) which vanishes at a and b and has exactly m zeros in (a, b). The possible presence of singular solutions does not permit an exact counterpart to this result. However, Theorem 3 does permit $a(t)$ to have zeros if solutions are unique.

THEOREM 3. *Suppose F satisfies* (5), (7), *and* (8) *and suppose all solutions of the initial value problem for* (1) *are unique on* (c, d), $0 \leqq c < d \leqq \infty$.

Then, for any interval $[a, b]$, $c<a<b<d$, *and any integer* $m\geqq 0$, *there is a solution of* (1) *with positive slope at a and a solution with negative slope at a each of which vanishes at a, b, and has exactly m zeros in* (a, b).

PROOF. Let $x(t, s, x_1)$ denote the solution of (1) passing through $(s, 0)$ with positive slope x_1. If $m=0$, Lemmas 1, 3, 5, and continuous dependence on x_1 ensure there is an x_1^0 such that the next zero of $x(t, a, x_1^0)$ is at $t=b$. If $m>0$ consider $\Delta=(2m+2)^{-1}(b-a)$. Then, for each s in $[a, b-2\Delta]$ there is an $x_1(s)>0$ such that the next zero of $x(t, s, x_1(s))$ is at $s+\Delta$. Continuous dependence on parameters implies there is for each s in $[a, b-2\Delta]$ an $\varepsilon(s)>0$ such that r in $(s-\varepsilon(s), s+\varepsilon(s))$ implies $x(t, r, x_1(s))$ has its next zero in $[s, s+2\Delta]$. Hence, compactness of $[a, b-2\Delta]$ ensures there is an x_1^+ such that if $0<x_1\leqq x_1^+$ then $x(t, s, x_1)$, $a\leqq s\leqq b-2\Delta$, has a first zero in $[s, s+2\Delta]$. Similarly, there is an x_1^- such that $0>x_1\geqq x_1^-$ implies $x(t, s, x_1)$, $a\leqq s\leqq b-2\Delta$, has a first zero in $[s, s+2\Delta]$. Since $x'(t, a, x_1)\rightarrow 0$ uniformly on $[a, b]$ as $x_1\rightarrow 0$ it is the result of an easy induction that $|x'(t, a, x_1^*)|\leqq\min(x_1^+, -x_1^-)$ implies $x(t, a, x_1)$ has at least $m+1$ zeros in $(a, b]$ for all $0<|x_1|\leqq x_1^*$. These zeros vary continuously with x_1 and in particular Lemma 5 implies there are a positive and a negative value for x_1 such that the $(m+1)$st zero is at $t=b$. This proves the theorem.

The next theorem is suggested by and improves a corollary which was given by Grimmer and Waltman [**4**].

THEOREM 4. *Under the conditions of Theorem* 3 *the equation* (1) *with boundary conditions* $x(a)=A$, $x(b)=B$, $0\leqq a<b<\infty$, $A\geqq B\geqq 0$, *has a unique positive solution.*

PROOF. If $A=0$, Theorem 3 ensures existence. If $A>0$, there is certainly an x_1 (possibly negative) such that $x(t, x_1)$ has a first zero in (a, b). Lemma 5 shows that this zero tends to infinity as x_1 tends to infinity. Hence, continuous dependence on x_1, convexity of solutions, and the Intermediate Value Theorem imply existence.

Uniqueness is a consequence of Theorem 1.

REMARK 2. The majorant condition (8) is only required for Lemma 5. Thus, it should be noted that if there is a solution $x(t, a, A, x_1)$ of (1) which is positive on (a, ∞) then, as in the first part of the proof of Lemma 5, it can be concluded that $T(x_1)\rightarrow\infty$ without recourse to the majorant condition.

5. **Oscillation.** A solution of (1) is said to be oscillatory if it has unbounded zeros. Belohorec [**2**] showed that all solutions of (2) are oscillatory if and only if $\int^\infty t^\gamma a(t)\,dt=\infty$. Theorem 5 permits the relating of such results to equations like (1). All solutions of the equations (10) and (11) are assumed to exist for all t-values to the right of some initial t-value.

THEOREM 5. *Given the equations*

$$x'' + F_1(t, x) = 0, \tag{10}$$

$$y'' + F_2(t, y) = 0, \tag{11}$$

and a function $F(t, u)$ which satisfies (5) *such that*

$$uF_1(t, u) \geqq uF(t, u) \geqq uF_2(t, u) \geqq 0,$$

$-\infty < u < \infty$. *Suppose F, F_1, F_2 are continuous on* $[0, \infty) \times (-\infty, \infty)$. *If all solutions of* (11) *are oscillatory then all solutions of* (10) *are oscillatory.*

PROOF. Suppose $x(t)$ solves (10) and $x(t)>0$ on $[T, \infty)$. Then, select a solution $y(t)$ of (11) such that $y(T)=x(T)$, $y'(T)>x'(T)$. Theorem 1 asserts $y(t)>x(t)$ on $[T, \infty)$. Since $y(t)$ oscillates this is a contradiction. The theorem is proved.

ACKNOWLEDGEMENTS. The author thanks J. W. Heidel for suggesting reference [3] and the referee for useful comments.

REFERENCES

1. P. B. Bailey, L. F. Shampine and P. E. Waltman, *Nonlinear two point boundary value problems*, Math. in Science and Engineering, vol. 44, Academic Press, New York, 1968. MR **37** #6524.

2. S. Belohorec, *Oscillatory solutions of certain nonlinear differential equations of the second order*, Mat.-Fyz. Časopis Sloven. Akad. Vied **11** (1961), 250–255.

3. ———, *Two remarks on the properties of solutions of a nonlinear differential equation*, Acta. Fac. Rerum Natur. Univ. Comenian. Math. **22** (1969), 19–26.

4. R. C. Grimmer and P. Waltman, *A comparison theorem for a class of nonlinear differential inequalities*, Monatsh. Math. **72** (1968), 133–136. MR **37** #3165.

5. P. Hartman, *Ordinary differential equations*, Wiley, New York, 1964. MR **30** #1270.

6. J. W. Heidel, *A nonoscillation theorem for a nonlinear second order differential equation*, Proc. Amer. Math. Soc. **22** (1969), 485–488. MR **40** #1648.

7. ———, *Uniqueness, continuation, and nonoscillation for a second order nonlinear differential equation*, Pacific J. Math. **32** (1970), 715–721. MR **41** #3886.

8. ———, *Rate of growth of nonoscillatory solutions for the differential equation* $y''+q(t)|y|^\gamma \operatorname{sgn} y=0$, $0<\gamma<1$, Quart. Appl. Math. **28** (1971), 601–606.

9. J. W. Heidel and D. B. Hinton, *The existence of oscillatory solutions for a nonlinear differential equation* (to appear).

10. R. A. Moore and Z. Nehari, *Nonoscillation theorems for a class of nonlinear differential equations*, Trans. Amer. Math. Soc. **93** (1959), 30–52. MR **22** #2755.

11. H. Teufel, Jr., *Estimation and extension for nonlinear oscillators* (to appear).

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